Contact mechanics I: basics

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4. Normal contact of inelastic solids
5. Contact of inhomogeneous bodies
Short historical sketch

Use and opposition to friction

- Frictional heat - lighting of fire - more than [40,000 years ago].
- Ancient Egypt - lubrication of surfaces with oil [5,000 years ago].
Short historical sketch

First studies on contact and friction

- Leonardo da Vinci [1452-1519]
  first friction laws and many other trobological topics;

- Issak Newton [1687]
  Newton’s third law for bodies interaction;

- Guillaume Amontons [1699]
  rediscovered friction laws;

- Leonhard Euler [1707-1783]
  roughness theory of friction;
First studies on contact and friction

- Charles-Augustin de Coulomb [1789]
  friction independence on sliding velocity and roughness; the influence of the time of repose.

- Heinrich Hertz [1881-1882]
  the first study on contact of deformable solids;

- Holm [1938], Ernst and Merchant [1940], Bowden and Tabon [1942]
  difference between apparent and real contact areas, adhesion theory.
**Practice VS theory**

- **1900**: Theory is several steps behind the practice

Theory  
Practice
1940: Theory is behind the practice
1960: Theory catches up with practice
1990: The trial-and-error testing becoming more and more difficult. Theory leads practice.
Plan

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3. Contact mechanics of elastic solids
4. Normal contact of inelastic solids
5. Contact of inhomogeneous bodies
Surface interaction properties

Surface properties:

- Coefficient of friction
- Adhesion
- Wear parameters
Surface properties are not fundamental

- Coefficient of friction 😞
- Adhesion 😞
- Wear parameters 😞
Surface interaction properties

Surface properties are not fundamental

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Fundamental properties:

- **Volume:**
  - Young’s modulus;
  - Poisson’s ratio;
  - shear modulus;
  - yield stress;
  - elastic energy;
  - thermal properties.

- **Surface:**
  - chemical reactivity;
  - absorption capabilities;
  - surface energy;
  - compatibility of surfaces;
Surface interaction properties

Surface properties **are not fundamental**

- Coefficient of friction 😞
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More fundamental properties

- solids are made of atoms;
- atoms are linked by bonds;
- many of the volume and surface properties are the properties of the bonds.

Fundamental properties are interdependent 😞

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Material properties interdependence

Young's modulus and yield strength interdependence [Rabinowicz, ]
Material properties interdependence

Penetration hardness and yield stress interdependence

Young’s modulus and melting temperature interdependence

[Rabinowicz, ]
Material properties interdependence

Thermal coefficient of expansion and Young's modulus interdependence [Rabinowicz, ]

Surface energy and hardness interdependence [Rabinowicz, ]
Real area of contact depends on

- **normal load:**
  real area of contact is proportional to the normal load; coefficient of proportionality is inverse of the material hardness;

- **sliding distance:**
  contact area might be $3(!)$ times as great as the value before shear forces were first applied;

- **time:** (for creeping materials)
  real area of contact increases with time;

- **surface energy:**
  the higher the surface energy, the greater the area of contact.

[Ref: Course of Julian Durand on surface roughness]
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\[ A_r \sim F \]

$A_r$ - real contact area, $F$ - applied load

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\[
A_r = \frac{F}{p}
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Engineering friction

First approximations: friction coefficient does not depend on

- normal load
- apparent area of contact
- velocity
- sliding surface roughness
- time

Friction force direction is opposite to the sliding
Engineering friction

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Real friction :: normal load

First approximation:
- friction coefficient does not depend on normal load.

Exceptions:
- at micro scale for small slidings (fig. 1);
- for very large normal loads (metal forming) friction force is limited;
- for very hard (diamond) or very soft (teflon) materials:
  - generally $T = cF^\alpha$, $\alpha \in \left[ \frac{2}{3}; 1 \right]$;
- thin hard coating and a softer substrate (fig.2).
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![Graph showing friction coefficient vs. normal load for steel on steel contact.](image)

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![Graph showing friction coefficient vs. load for copper on copper contact.](image)

**Fig. 2.** In case of hard surface layer on a softer substrate, at moderate loads friction is determined by the hard surface, higher load brakes the coating and softer material begins to define the frictional properties [Rabinowicz, ]
Real friction :: normal force

Friction coefficient versus tangential movement; experiments from [Courtney-Pratt and Eisner, 1957]
Real friction :: friction direction

First approximation:

- friction force direction is opposite to the sliding.

Exceptions:

- the direction of the friction force remains within $[178; 182]$ degrees to sliding direction (fig. 1);
- the difference is higher for oriented surface roughnesses.
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Fig. 1. Change of the direction of friction force with sliding

[Rabinowicz, ]
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[Rabinowicz, ]
Real friction :: apparent area and roughness

First approximation:
- Friction coefficient does not depend on the apparent area of contact.

Exceptions:
- very smooth and very clean surfaces.

First approximation:
- Friction coefficient does not depend on sliding surface roughness.

Exceptions:
- very smooth or very rough surfaces (fig. 1).
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Fig. 1. Friction roughness influences the coefficient of friction [Rabinowicz, ]
First approximation:
- Friction coefficient does not depend on time.

Exceptions:
- creeping materials.

First approximation:
- Friction coefficient does not depend on sliding velocity.

Exceptions:
- if material behaves differently at different loading rate, then the friction depends on the sliding velocity;
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Exceptions:
- If material behaves differently at different loading rate, then the friction depends on the sliding velocity.

$$f_s = f_0 + kt^{1/10}$$

Static friction evolution with time.
Real friction :: time and velocity

First approximation:

- Friction coefficient does not depend on time.

Exceptions:

- Creeping materials.

First approximation:

- Friction coefficient does not depend on sliding velocity.

Exceptions:

- If material behaves differently at different loading rate, then the friction depends on the sliding velocity.

\[ f_s = f_0 + kt^{1/10} \]

\[ f_K = f_d + (f_s - f_d)e^{cv} \]
Real friction :: velocity

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- Friction coefficient does not depend on sliding velocity.

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Friction coefficient slightly decreases with increasing velocity of sliding, titanium on titanium [Rabinowicz, ]
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Friction coefficient dependence on velocity of sliding for lubricated surfaces [Rabinowicz, ]
Real friction :: velocity

First approximation:
- Friction coefficient does not depend on sliding velocity.

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- if material behaves differently at different loading rate, then the friction depends on the sliding velocity;

Friction coefficient increases and decreases with increasing velocity of sliding, hard on soft (steel on lead, steel on indium) [Rabinowicz, ]

Friction coefficient dependence on velocity of sliding for lubricated surfaces [Rabinowicz, ]
Three scales of contact study

**Nanoscale:**
Study of molecular junctions, van des Waals forces and Casimir effect.

**Microscale:**
Roughness and microstructure study

**Macroscale:**
Stress-strain state of contacting solids
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Macroscopic contact

- Signorini contact law (1933)

\[
\begin{cases}
F_n \leq 0 \\
u_n \leq 0 \\
F_n u_n = 0
\end{cases}
\]

- Compliance contact law

\[F_n = C_n(u_n)^{m_n}\]

\[F_n = C_1 e^{c_2 u_n^2}\]

[Kragelsky, 1982], [Oden-Martins, 1985]

[Song, Yovanovich, 1987]
Hertz theory (1882)

Geometry of smooth, non-conforming surface in contact

- Expression of the profile of each surface

\[
\begin{align*}
z_1 &= \frac{1}{2R'_1}x_1^2 + \frac{1}{2R''_1}y_1^2 \\
z_2 &= -\left(\frac{1}{2R'_2}x_2^2 + \frac{1}{2R''_2}y_2^2\right)
\end{align*}
\]

where \( R'_i \) and \( R''_i \) are the principal radii of curvature of the surface \( i \).

- Separation between the two surfaces

\[
h = z_1 - z_2 = Ax^2 + By^2
\]

- Displacement

\[
\bar{u}_{z_1} + \bar{u}_{z_2} + h = \delta_1 + \delta_2
\]

[Johnson, 1996]
Hertz theory (1882)

Assumptions in the Hertz theory:

- The surface are continuous and non-conforming, \( a \ll R \)
- The strains are small, \( a \ll R \)
- Each solid can be considered as an elastic half-space, \( a \ll R_{1,2}, a \ll l \)
- The surfaces are frictionless, \( q_x - q_y = 0 \)

Applications

1. Solids of revolution
2. Two-dimensional contact of cylindrical bodies

Note

\[
\frac{1}{E^*} = \frac{1 - \nu_1}{E_1} + \frac{1 - \nu_2}{E_2}
\]
**Hertz theory : Solids of revolution**

*Simple case of solids of revolution*

- **Principal radii of curvature**
  \[ R'_i = R''_i = R_i, \quad i = 1, 2 \]

- **Boundary conditions for the displacement**
  \[ \overline{u_{z1}} + \overline{u_{z2}} = \delta - \left(\frac{1}{2R}\right)r^2 \]

- **Pressure distribution**
  \[ p = p_0 \left\{ 1 - \left(\frac{r}{a}\right)^2 \right\}^{1/2} \]

**Consequences**

- **Pressure**
  \[ p_0 = \left(\frac{6PE^*2}{P^3R^2}\right)^{1/3} \]

- **Radius of the contact circle**
  \[ a = \left(\frac{3PR}{4E^*2}\right)^{1/3} \]

- **Displacement**
  \[ \delta = \left(\frac{9P^2}{16RE^*2}\right)^{1/3} \]
Hertz theory : Solids of revolution

\[ \frac{\sigma_r}{p_0}(x = 0, z) = \frac{\sigma_\theta}{p_0}(x = 0, z) = -(1 + \nu) \left\{ 1 - \left(\frac{z}{a}\right) \tan^{-1}\left(\frac{a}{z}\right) \right\} + \frac{1}{2} \left(1 + \frac{z^2}{a^2}\right)^{-1} \]

\[ \frac{\sigma_z}{p_0}(x = 0, z) = -(1 + \frac{z^2}{a^2})^{-1} \]

Maximum shear stress \( \tau_1 = \frac{1}{2} |\sigma_r - \sigma_\theta| \)  
(\( \tau_1 \))_{max} = 0.31p_0 at the depth of 0.48a (for \( \nu = 0.3 \))
2D contact of cylindrical bodies

\[ p(x) = p_0 (1 - (x/a)^2)^{1/2} \]

\[ a = \sqrt{\frac{4PR}{\pi E^*}} \]

\[ p_0 = \sqrt{\frac{PE^*}{\pi R}} \]

\[ \frac{1}{E^*} = \frac{1 - \nu_1}{E_1} + \frac{1 - \nu_2}{E_2} \]

\[ \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \]
2D contact of cylindrical bodies

*Example: cylinder/plate*

Distributions of normal pressure (Hertz) and tangential stress

\[
\begin{align*}
\sigma_{xx}(x = 0, z) &= -\frac{p_0}{a} \left\{ (a^2 + 2z^2)(a^2 + z^2)^{-1/2} - 2z \right\} \\
\sigma_{zz}(x = 0, z) &= -p_0 a (a^2 + z^2)^{-1/2} \\
\tau_{\text{max}}(x = 0, z) &= p_0 a \left\{ z - z^2(a^2 - z^2)^{-1/2} \right\} \\
\sigma_{xz} &= \sigma_{xy} = \sigma_{yz} = 0
\end{align*}
\]
Macroscopic friction 1/2

- **Tresca**

\[ |F_t| \leq g \]

If \( |F_t| < g \), then \( V_{slide} = 0 \)

If \( |F_t| = g \), \( \exists \lambda > 0 \) such that \( V_{slide} = -\lambda F_t \)

- **Coulomb**

\[ |F_t| \leq \mu |F_n| \]

If \( |F_t| < \mu |F_n| \), then \( V_{slide} = 0 \) stick

If \( |F_t| = \mu |F_n| \), \( \exists \lambda > 0 \) such that \( V_{slide} = -\lambda F_t \) slip
Regularized Coulomb [Oden, Pires, 1983], [Raous, 1999]

\[ |F_t| = -\mu \phi^i (V_{\text{slide}}) |F_n| \]
\[ \phi^1 = \frac{V_{\text{slide}}}{\sqrt{V_{\text{slide}}^2 + \varepsilon^2}} \]
\[ \phi^2 = \tanh \left( \frac{V_{\text{slide}}}{\varepsilon} \right) \]

Variable friction

\[ |F_t| \leq C_t (u_t)^{m_t} \]
\[ \begin{align*}
    \text{If } |F_t| < C_t (u_t)^{m_t}, \text{ then } V_{\text{slide}} = 0 \\
    \text{If } |F_t| = C_t (u_t)^{m_t}, \exists \lambda > 0 \text{ such } V_{\text{slide}} = -\lambda R_t
\end{align*} \]
Introduction

Basic knowledges

Elastic contact

Inelastic contact

Contact of composites

Transition toward the slip

**Definition of sliding**
Relative peripheral velocity of the surfaces at their point of contact

**Sliding of non-conforming elastic bodies**

- **Question**
  The tangential traction due to the friction at the contact surface influences the size and shape of the contact area or the distribution of normal pressure?

- **Evaluation of the elastic stresses and displacements**
  Basic premise of the Hertz theory

- **Relationship between the tangential traction and the normal pressure**

\[
\frac{|q(x, y)|}{p(x, y)} = \frac{|Q|}{P} = \mu
\]
Coulomb's law

- **Application**
  Cylinder sliding perpendicular to its axis

[Johnson, 1996, Goryacheva, 1998]
Cylinder sliding perpendicular to its axis

Distributions of normal pressure (Hertz) and tangential traction

\[ p(x) = \frac{2P}{\pi a} \sqrt{1 - \left(\frac{x}{a}\right)^2} \]
\[ q(x) = \pm \mu p(x) = \pm \mu \frac{2P}{\pi a} \sqrt{1 - \left(\frac{x}{a}\right)^2} \]  

(1)

Stress components

\[ \sigma_{xx}(x, z = 0) = -p_0 \left\{ \sqrt{1 - \left(\frac{x}{a}\right)^2} + 2 \mu \frac{x}{a} \right\} \]
\[ \sigma_{zz}(x, z = 0) = -p_0 \sqrt{1 - \left(\frac{x}{a}\right)^2} \]
\[ \sigma_{yy}(x, z = 0) = 0 \]
\[ \tau_{xz}(x, z = 0) = -\mu p_0 \sqrt{1 - \left(\frac{x}{a}\right)^2} \]
\[ \tau_{xy}(x, z = 0) = \tau_{yz}(x, z = 0) = 0 \]
Partial slip

Relation between slip zone \((c)\) and contact zone \((a)\)

\[
q_1(x) = \mu p_0 \left(1 - \frac{x^2}{a^2}\right)^{1/2}
\]

\[
q_2(x) = \mu \frac{c}{a} p_0 \left(1 - \frac{x^2}{c^2}\right)^{1/2}
\]

\[
q(x) = q_1(x) - q_2(x)
\]

\[
\frac{c}{a} = \sqrt{1 - \frac{Q}{\mu P}}
\]

- If \(x < c\) : stick condition. The local contact shear stress is

\[
\tau_{xz} = \mu p_0 \sqrt{1 - \left(\frac{x}{a}\right)^2} - \frac{c}{a} \mu p_0 \sqrt{1 - \left(\frac{x}{c}\right)^2}
\]

- If \(c < x < a\): slip condition. The local contact shear stress is

\[
\tau_{xz} = \mu p_0 \sqrt{1 - \left(\frac{x}{a}\right)^2}
\]
**Definition of the stick-slip**: Intermittent relative motion between the contact surfaces, alternation of slip and stick.

Phenomenon occurs at various scales:

- **Macroscopic**: discontinuities in the gravity center displacement of contact body and loads.
- **Microscopic**: location of the phenomenon at the interface
The stick-slip is a coupling result:

- **The dynamic response of the friction system**
  stiffness, damping, inertia

- **Friction dynamic at the interface**
  Difference between static ($\mu_s$) and dynamic ($\mu_d$) friction coefficient
  $\mu_s$ and $\mu_d$ dependence on the sliding velocity and time

**A simple stick-slip model**

![Friction law](image)

Fig: Plot of frictional force vs. time.
During sliding, the problem is:

\[ m\ddot{x} - F_d = -kx \quad x(0) = \frac{F_s}{k} \quad \dot{x}(0) = v \]

and the solution is

\[ x(t) = \frac{1}{k} \left\{ (F_s - F_d)\cos(\omega t) + F_d \right\} + \frac{v}{\omega} \sin(\omega t), \quad \omega = \sqrt{\frac{k}{m}} \]

or the velocity \( v \) is negligible compared to \( \frac{dx}{dt} \):

\[ x(t) \approx \frac{1}{k} \left\{ (F_s - F_d)\cos(\omega t) + F_d \right\} \]

Characteristic time of sliding

\[ T_{\text{inertia}} = 2\pi \sqrt{\frac{m}{k}} \]

The force \( F \) oscillates between \( F_s \) and \( 2F_d - F_s \).
Friction instability: “stick-slip” 4/4

The red curve in the parameters plane at the other parameters being fixed, demarcates the regions of stable and unstable motion.

Fig: Regions of stable and stick-slip motion.
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Examples

(a) Vickers indentation test, palladium glasses

(b) Contact zone under Vickers indenter, zirconium glasses

(c) Scratch resistance of soda-Lime Silica Glasses
Hill’s theory: Elastic-plastic indentation

**Cavity model of an elastic-plastic indentation cone**

**Assumptions**

- Within the core:
  
  Hydrostatic component of stress $\bar{p}$

- Outside the core:
  
  Radial symmetry for stresses and displacement

- At the interface (between core and plastic zone)
  
  Hydrostatic stress (in the core) = radial component of stress (in the external zone)

- The radial displacement on $r=a$ during an increment $dh$ must accommodate the volume of material.
**Characteristic**

In the plastic zone: \( a \leq r \leq c \)
\[
\frac{\sigma_r}{Y} = -2\ln\left(\frac{c}{r}\right) - \frac{2}{3}
\]
\[
\frac{\sigma_\theta}{Y} = -2\ln\left(\frac{c}{r}\right) + \frac{1}{3}
\]

In the elastic zone: \( r \geq c \)
\[
\frac{\sigma_r}{Y} = -\frac{2}{3} \left( \frac{\varepsilon_r}{r} \right)^3
\]
\[
\frac{\sigma_\theta}{Y} = \frac{1}{3} \left( \frac{\varepsilon_r}{r} \right)^3
\]

where \( Y \) denotes the value of the yield stress of material in simple shear and simple compression.

Core pressure
\[
\bar{p} \frac{Y}{Y} = -\left[ \frac{\sigma_r}{Y} \right]_{r=a} = \frac{2}{3} + 2\ln\left(\frac{c}{a}\right)
\]

Radial displacement
\[
\frac{du(r)}{dc} = \frac{T}{E} \left\{ 3(1 - \nu)(c^2/r^2) - 2(1 - 2\nu)(r/c) \right\}
\]

Conservation volume
\[
2\pi a^2 du(a) = \pi a^2 dh = \pi a^2 \tan(\beta) da
\]
Unloading indentation: elastic strain energy

Example: spherical indenter

Before loading

Under loading

After unloading

Residual depth $\delta - \delta'$: Estimation of the energy dissipated $\Delta W$ in one cycle of the load

$\Delta W = \alpha \int P d\delta$ where $\alpha$ is the hysteresis-loss factor. ($\alpha = 0, 4\%$ for hard bearing steel)

$W = \frac{2}{5} \left( \frac{9E^*^2 P^5}{16R} \right)$

$4a^3 \left( \frac{1}{R} - \frac{1}{\rho} \right) = \frac{3P}{E^*}$

$a = \left( \frac{3P}{4E'} \right)^{1/3}$

$\delta = \frac{a^2}{R} = \left( \frac{9P^2}{16RE'/3} \right)^{1/3}$

$\delta' = \frac{9\pi P_p m}{16E'/2}$ with $P_m = 0.38Y$ in fully plastic state

$\frac{P}{P_Y} = 0, 38(\delta'/\delta_Y)^2$
Introduction

Basic knowledges

Elastic contact

Inelastic contact

Contact of composites

Sharp indentation

Characterisation of $P-h$ response

- During the loading,

\[ P = Ch^2 \quad \text{Kick's law} \]

- During the unloading,

\[ \left. \frac{dP_u}{dh} \right|_{h_m} \quad \text{initial slope} \]

\[ h_r \quad \text{Residual indentation depth after complete unloading} \]

Three independent quantities
Plastic behavior

Model

\[
\sigma = \begin{cases} 
E \varepsilon & \text{for } \sigma \leq \sigma_y \\
R \varepsilon^n & \text{for } \sigma > \sigma_y 
\end{cases}
\]

- E: Young's modulus
- R: a strength coefficient
- n: the strain hardening exponent
- \( \sigma_y \): the initial yield stress

Assumption:
The theory of plasticity with the von Mises effect stress.

Parameters for an elasto-plastic behavior
E, \( \nu \), \( \sigma_y \), n
**Objective**
Prediction of the $P - h$ response from elasto-plastic properties

Application of the universal dimensionless functions : the $\Pi$ theorem

**Material parameter set**

$$(E, \nu, \sigma_y, n) \quad \text{or} \quad (E, \nu, \sigma_r, n) \quad \text{or} \quad (E, \nu, \sigma_y, \sigma_r)$$

- **Load $P$**

$$P = P(h, E^*, \sigma_y, n) \quad \text{or} \quad P = P(h, E^*, \sigma_r, n) \quad \text{or} \quad P = P(E, \nu, \sigma_y, \sigma_r)$$

with

$$E^* = \left( \frac{1 - \nu^2}{E} + \frac{1 - \nu_i^2}{E_i} \right)^{-1}$$

- **Unload**

$$P_u = P_u(h, h_m, E, \nu, E_i, \nu_i, \sigma_r, n) \quad \text{or} \quad P_u = P_u(h, h_m, E^*, \sigma_r, n)$$
Determination of $h_m$

Application of the dimensional analysis during the load

<table>
<thead>
<tr>
<th>Load</th>
<th>Applying the Π theorem in dimensional analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P = P(h, E^*, \sigma_y, n)$</td>
<td>$C = \frac{P}{h^2} = \sigma_r \Pi_1 \left( \frac{E^*}{\sigma_r}, n \right)$</td>
</tr>
<tr>
<td>$P = P(h, E^*, \sigma_r, n)$</td>
<td>$C = \frac{P}{h^2} = \sigma_y \Pi_1^A \left( \frac{E^*}{\sigma_y}, \frac{\sigma_r}{\sigma_y} \right)$</td>
</tr>
<tr>
<td>$P = P(E, \nu, \sigma_y, \sigma_r)$</td>
<td>$C = \frac{P}{h^2} = \sigma_r \Pi_1^B \left( \frac{E^*}{\sigma_r}, \frac{\sigma_y}{\sigma_r} \right)$</td>
</tr>
</tbody>
</table>

with $\Pi$ are dimensionless functions.

And then $h_m$!
Determination of $h_r$

Application of the dimensional analysis during the unload

$$
\frac{dP_u}{dh} = \frac{dP_u}{dh} (h, h_m, E^*, \sigma_r, n)
$$

thus

$$
\frac{dP_u}{dh} = E^* h \Pi_2^0 \left( \frac{h_m}{h}, \frac{\sigma_r}{E^*}, n \right)
$$

Consequently,

$$
\left. \frac{dP_u}{dh} \right|_{h=h_m} = E^* h_m \Pi_2^0 \left( 1, \frac{\sigma_r}{E^*}, n \right) = E^* h_m \Pi_2 \left( \frac{\sigma_r}{E^*}, n \right)
$$

Or

$$
P_u = P_u (h, h_m, E^*, \sigma_r, n) = E^* \Pi_u \left( \frac{h_m}{h}, \frac{\sigma_r}{E^*}, n \right)
$$

Finally,

$$
P_u = 0 \text{ implies } 0 = \Pi_u \left( \frac{h_m}{h_r}, \frac{\sigma_r}{E^*}, n \right) \text{ whether } \frac{h_r}{h_m} = \Pi_3 \left( \frac{\sigma_r}{E^*}, n \right)
$$

and then $h_r$ !

$\Pi_{i,i = 1,2,3}$ can be used to relate the indentation response to mechanical properties.
FE Vickers indentation test

Maximum penetration $h_{s}^{\text{max}}$
3.11 $\mu$m

Parameters

Elastic

E=81600MPa, $\nu = 0, 36$

Elasto-plastic model

E=81600MPa, $\nu = 0, 36$, 
$\sigma_y = 1610$MPa

Drucker-Prager model

E=81600MPa, $\nu = 0, 36$, 
$\sigma_y = 1600$MPa, $\sigma_y^c = 1800$MPa

[Laniel, 2004]
Computation results

Penetration depth

\[ \text{von Mises}_{\text{max}} \]

Residual stress
Elasto-plastic contact response

![Elasto-plastic contact response graph](image)
Elasto-plastic contact response
**Hypothesis**
Sphere radius: \( R = 100 \mu m \)
Copper and zinc single crystals: crystal plasticity
Silicon substrate: isotropic elastic
Maximum penetration \( h_{s}^{\text{max}} \): 3.5 \( \mu m \)

[Casal and Forest, 2009]
Contact response

(d) Elastic anisotropic contact response
(e) Elasto-plastic anisotropic contact response
von Mises stress fields

**Figure:** (a) f.c.c and (b) h.c.p crystals. Penetration depth: $h_s = 1.25\mu m$
Plastic zone morphology

(a) f.c.c copper crystals
(b) h.c.p zinc crystals

Figure: Penetration depth: $h_s = 3.5 \mu m$
Plan

1. Introduction
2. Basic knowledges
3. Contact mechanics of elastic solids
4. Normal contact of inelastic solids
5. Contact of inhomogeneous bodies
Bounds for the global coefficient of friction

\[ \mu(x) = \frac{q(x)}{p(x)} \quad \bar{\mu} = \frac{\int \mu(x)p(x)dS}{\int p(x)dS} \]

Uniform stress \( \equiv \sum \mu_i f_i \leq \mu \leq \frac{\sum \mu_i c_i f_i}{\sum c_i f_i} \equiv \) Uniform strain

with \( c_i = \frac{E_i}{(1-\nu_i^2)} \frac{(1-\nu_i)^2}{(1-2\nu_i)} \)

[Dick and Cailletaud, 2006]
Bounds for the global coefficient of friction

<table>
<thead>
<tr>
<th>Cont A</th>
<th>ν</th>
<th>E (GPa)</th>
<th>C (GPa)</th>
<th>µ</th>
<th>Cont B</th>
<th>ν</th>
<th>E (GPa)</th>
<th>C (GPa)</th>
<th>µ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp 1</td>
<td>0.32</td>
<td>8</td>
<td>11.45</td>
<td>0.1</td>
<td>Comp 1</td>
<td>0.15</td>
<td>55</td>
<td>58.08</td>
<td>0.1</td>
</tr>
<tr>
<td>Comp 2</td>
<td>0.15</td>
<td>55</td>
<td>58.08</td>
<td>0.5</td>
<td>Comp 2</td>
<td>0.32</td>
<td>08</td>
<td>11.45</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Case A: \( \mu_1 = 0.1\), \( E_1 = 11.45\, \text{GPa} \)
\( \mu_2 = 0.5\), \( E_2 = 58\, \text{GPa} \)

Case B: \( \mu_1 = 0.5\), \( E_1 = 58\, \text{GPa} \)
\( \mu_2 = 0.1\), \( E_2 = 11.45\, \text{GPa} \)
FE computations VS analytic estimation

![Graph showing COF vs Component 2 (%) and normalized contact pressure vs x (mm)]
Different CSL geometries 1/2

<table>
<thead>
<tr>
<th></th>
<th>Bulk material</th>
<th>Component 1</th>
<th>Component 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$ (GPa)</td>
<td>119</td>
<td>8</td>
<td>55</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.29</td>
<td>0.32</td>
<td>0.15</td>
</tr>
<tr>
<td>$R_0$ (MPa)</td>
<td>-</td>
<td>200</td>
<td>500</td>
</tr>
</tbody>
</table>
Different CSL geometries 2/2

- Introduction
- Basic knowledges
- Elastic contact
- Inelastic contact
- Contact of composites

Diagram showing COF (Coefficient of Friction) versus Component 2 (%) for different contact pressures (normalized contact pressure vs. x (mm)) for components 1 and 2.
Influence of the number of dimples

<table>
<thead>
<tr>
<th>ESD type</th>
<th>2a (µm)</th>
<th>$l_{ESD}$ (µm)</th>
<th>nb. ESD</th>
<th>$l_{ESD}$ (µm)</th>
<th>nb. ESD</th>
</tr>
</thead>
<tbody>
<tr>
<td>R10</td>
<td>360</td>
<td>11.1</td>
<td>32.4</td>
<td>33.3</td>
<td>10.8</td>
</tr>
<tr>
<td>R20</td>
<td>360</td>
<td>22.2</td>
<td>16.2</td>
<td>66.6</td>
<td>5.4</td>
</tr>
<tr>
<td>R40</td>
<td>360</td>
<td>44.4</td>
<td>8.1</td>
<td>133.3</td>
<td>2.7</td>
</tr>
</tbody>
</table>
Influence of plastic deformations

- Basic knowledges
- Elastic contact
- Inelastic contact
- Contact of composites

- COF

- Component 2 (%)
- Component 1

- R40_el: p Comp.1, p Comp.2
- R40_pl: p Comp.1, p Comp.2

- Normalized contact pressure
- x (mm)

- C Comp.1, C Comp.2

- G. Cailletaud, S. Basseville, V.A. Yastrebov — MINES ParisTech, UVSQ

Contact mechanics I
Paris, 21-24 June 2010
Estimation of the upper and lower bound.

The friction coefficient depend on

- the CSL geometry
- the dissimilarity of the CSL component materials
- the compliance of substrate and counter body
Summary

- Contact and friction
  - complicated phenomena;
  - depend on many material properties;
  - not yet well elaborated.

- Analytical solutions
  - hertzian contact;
  - nonlinear material;
  - friction;
  - stick-slip instabilities.

- Numerical analysis
  - examples of indendation tests;
  - analysis of heterogeneous friction.
Thank you for your attention!

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Stéphanie Basseville <stephanie.basseville@mines-paristech.fr>
Vladislav A. Yastrebov <vladislav.yastrebov@mines-paristech.fr>
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*Friction and Wear.*