Introduction to
Computational Contact Mechanics

Part I. Basics

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WEMESURF short course on contact mechanics and tribology
Paris, France, 21-24 June 2010
Preface

- To whom the course is aimed?
- Developers and users.
- What is the aim?
- Accurate contact modeling, correct interpretation, etc.
- FEM - Finite Element Method, FEA - Finite Element Analysis
Outline

- Mathematical foundation
  - contact geometry;
  - optimization methods.

- Inside the Finite Element programme
  - contact detection;
  - contact discretization;
  - account of contact.

- Contact problem resolution with FEM: guide for engineer
  - master-slave approach;
  - boundary conditions;
  - spurious cases.

- Content
  - demonstrative;
  - simple;
  - general.
Plan

1. Introduction
2. Contact detection
3. Contact geometry
4. Contact discretization methods
5. Solution of contact problem
6. Finite Element Analysis of contact problems
7. Numerical examples
Finite Element Method

- FEM a powerfull and multi-purpose tool for linear and non-linear dynamic and static continuum mechanical and multi-physic problems

[Rousselier et al.]
[Klyavin et al.]
[Brinkmeiera et al.]
Finite Element Method

- FEM a powerful and multi-purpose tool for linear and non-linear dynamic and static continuum mechanical and multi-physic problems.

- Contact problems are not continuum and they require:
  - new rigorous mathematical basis: geometry, optimization, non-smooth analysis;
  - particular treatment of the finite element algorithms;
  - smart use of the Finite Element Analysis (FEA).
Finite element analysis of contact problems

- assembled components;

Disk-blade contact

T. Dick, G. Cailletaud
Centre des Matériaux
Finite element analysis of contact problems

- assembled components;
- bearings;

Rolling bearing

F. Massi et al.
LaMCoS, INSA-Lyon et al.
Finite element analysis of contact problems

- assembled components;
- bearings;
- rolling contact;

Tire rolling noise simulation

M. Brinkmeier, U. Nackenhorst et al.
University of Hannover et al.
Finite element analysis of contact problems

- assembled components;
- bearings;
- rolling contact;
- forming processes;

Deep drawing

G. Rousselier et al.
Centre des Matériaux et al.
Finite element analysis of contact problems

- assembled components;
- bearings;
- rolling contact;
- forming processes;
- geomechanical contact;

Post seismic relaxation

J.D. Garaud, L. Fleitout, G. Cailletaud
Centre des Matériaux, ENS
Finite Element Method

Finite element analysis of contact problems

- assembled components;
- bearings;
- rolling contact;
- forming processes;
- geomechanical contact;
- crash tests;

Crash-test

O. Klyavin, A. Michailov, A. Borovkov
St Petersburg State University
Finite element analysis of contact problems

- assembled components;
- bearings;
- rolling contact;
- forming processes;
- geomechanical contact;
- crash tests;
- human joints;

Knee simulation

E. Peña, B. Calvoa et al.
University of Zaragoza
Finite element analysis of contact problems

- assembled components;
- bearings;
- rolling contact;
- forming processes;
- geomechanical contact;
- crash tests;
- human joints;
- and many others.

Knee simulation

E. Peña, B. Calvoa et al.
University of Zaragoza
They trust in the FEA
Leading international industrial companies
Mechanical problem
From a real life problem to an engineering problem

Need to determine:

- the **problematic**:
  - strength/life-time/fracture;
  - vibration/buckling;
  - thermo-electro-mechanical.

- relevant **geometry**;

- relevant **loads**:
  - static/quasi-static/dynamic;
  - mechanical/thermic;
  - volume/surface.

- relevant **material**:
  - rigid/elastic/plastic/visco-plastic;
  - brittle/ductile.

- relevant **scale**:
  - macro/meso/micro.
Mechanical problem
From a real life problem to an engineering problem

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**For example: Brinell hardness test**

Scheme of the Brinell hardness test and different types of impression

[Harry Chandler, Hardness testing]
Mechanical problem
From a real life problem to an engineering problem

Need to determine:

- **problematic:**
  - strength.
- **geometry:**
  - sphere + half-space.
- **loads:**
  - mechanical surface quasi-static.
- **material:**
  - rigid + elasto-visco-plastic.
- **scale:**
  - macro.

For example: Brinell hardness test

Scheme of the Brinell hardness test and different types of impression

[Harry Chandler, Hardness testing]
Mechanical problem
From an engineering problem to a finite element model

Problem:
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FE model:
- **analysis type:**
  - stress-strain state;
  - eigen values;
  - coupled physics.
Mechanical problem
From an engineering problem to a finite element model

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- **finite element mesh:**
Mechanical problem
From an engineering problem to a finite element model

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  - volume/surface.

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- **analysis type:**
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- **analysis type and BC:**
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Mechanical problem
From an engineering problem to a finite element model

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  - rigid/elastic/plastic/visco-plastic;
  - brittle/ductile.
- **microstructure:**
  - RVE/microstructure/—.
Mechanical problem
From engineering problem to a finite element model

Problem:
- **problematic:** strength.
- **geometry:**
- **loads:**
- **material:**
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FE model:
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Mechanical problem
From engineering problem to a finite element model

Problem:
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FE model:
- **analysis type:**
  - stress-strain state.
- **finite element mesh:**
  - sphere + large block.
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Problem:

- **problematic:** strength.
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FE model:

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- **material model:**
- **microstructure:**
Mechanical problem
From engineering problem to a finite element model

Problem:

- **problematic:**
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FE model:

- **analysis type:**
  - stress-strain state.
- **finite element mesh:**
  - sphere + large block.
- **analysis type and BC:**
  - mechanical surface quasi-static.
- **material model:**
  - rigid\(^a\) + elasto-visco-plastic model.
- **microstructure:**

\(^a\)rigid in FEA: much more harder than another solid, special boundary conditions or geometrical representation.
Mechanical problem
From engineering problem to a finite element model

Problem:
- **problematic:**
  - strength.
- **geometry:**
  - sphere + half-space.
- **loads:**
  - mechanical surface quasi-static.
- **material:**
  - rigid + elasto-visco-plastic.
- **scale:**
  - macro.

FE model:
- **analysis type:**
  - stress-strain state.
- **finite element mesh:**
  - sphere + large block.
- **analysis type and BC:**
  - mechanical surface quasi-static.
- **material model:**
  - rigid + elasto-visco-plastic model.
- **microstructure:**
  - homogeneous > RVE.

\[\text{rigid in FEA: much more harder than another solid, special boundary conditions or geometrical representation.}\]
Mechanical problem
From engineering problem to a finite element model

Problem:
- problematic
- geometry
- loads
- material
- scale

FE model:
- analysis type
- finite element mesh
- analysis type and BC
- material model
- microstructure

- Elasto-visco-plastic
- Potential contact zone
- Rigid
Mechanical problem
From engineering problem to a finite element model

Problem:
- problematic
- geometry
- loads
- material
- scale

FE model:
- analysis type
- finite element mesh
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- material model
- microstructure

Finite element mesh : )
Symmetry and plane problems
How to solve problems faster

Main ideas:
- symmetry
  - geometry AND loading;
- 3D to 2D:
  - axisymmetry/plane
    strain/plane stress;
- 3D to smaller 3D:
  - half/quarter/sector;
Symmetry and plane problems
How to solve problems faster

Main ideas:
- symmetry
  - geometry AND loading;
- 3D to 2D:
  - axisymmetry/plane strain/plane stress;
- 3D to smaller 3D:
  - half/quarter/sector;

Axisymmetry

Axisymmetry of geometry
axisymmetry of load
Symmetry and plane problems
How to solve problems faster

Main ideas:
- symmetry
  - geometry AND loading;
- 3D to 2D:
  - axisymmetry/plane strain/plane stress;
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Mirror symmetry
Axisymmetry of geometry
mirror symmetry of load
Symmetry and plane problems
How to solve problems faster

Main ideas:
- symmetry
  - geometry AND loading;
- 3D to 2D:
  - axisymmetry/plane strain/plane stress;
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No symmetry

Axisymmetry of geometry
no symmetry of load
Symmetry and plane problems
How to solve problems faster

Main ideas:
- symmetry
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- 3D to 2D:
  - axisymmetry/plane strain/plane stress;
- 3D to smaller 3D:
  - half/quarter/sector;

Plain strain

Mirror symmetry of geometry
mirror symmetry of load
very long structure or fixed edges
**Symmetry and plane problems**
How to solve problems faster

**Main ideas:**
- symmetry
  - geometry AND loading;
- 3D to 2D:
  - axisymmetry/plane strain/plane stress;
- 3D to smaller 3D:
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**Plain stress**

**Mirror symmetry of geometry**

**Mirror symmetry of load**

**Very thin structure**
Symmetry and plane problems
How to solve problems faster
Finite element mesh
Basics

In general the finite element mesh should

- fulfil the required solution precision;
- correctly represent relevant geometry;
- not be enormous;
- be fine where strain is large;
- be rough where strain is small;
- avoid too oblong elements.

In contact problems the finite element mesh should

- not allow corners at master surface;
- be very fine and precise on both contacting surfaces;
- use carefully quadratic elements.
Boundary conditions

- Two solids $\Omega_1$ and $\Omega_2$.
- Volumetric forces $F_v$: e.g. inertia $m\ddot{u}$.
- Neumann (static, force) boundary conditions: distributed and concentrated loads.
- Dirichlet (kinematic, displacement) boundary conditions: displacements.
- In each solid we fulfil $\text{div}(\sigma) + F_v = 0$.
Boundary conditions

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$\text{div}(\mathbf{S}) + F_v = 0$

$\mathbf{S}\cdot \mathbf{n} = \mathbf{S}_n$

$\mathbf{U} = \mathbf{U}_0$
**Boundary conditions**

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- What are the contact boundary conditions?
Boundary conditions

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- What are the contact boundary conditions?
Account of contact
Signorini conditions

- Signorini conditions of nonpenetration $g_n > 0$ and non-adhesion $\sigma_n \leq 0$

$$g_n \sigma_n = 0, \quad g_n \geq 0, \quad \sigma \leq 0, \quad \sigma_n = \sigma \cdot n$$

- Contact boundary conditions $\sim$ unknown Neumann (force) boundary conditions depending on the geometry.
Signorini conditions of nonpenetration $g_n > 0$ and non-adhesion $\sigma_n \leq 0$

\[ g_n\sigma_n = 0, \quad g_n \geq 0, \quad \sigma \leq 0, \quad \sigma_n = \sigma \cdot n \]

Contact boundary conditions $\sim$ unknown Neumann (force) boundary conditions depending on the geometry.
Account of contact
Coulomb’s friction

Coulomb’s friction conditions

\[ |\dot{g}_t| (|\sigma_t| + \mu \sigma_n) = 0; \quad |\sigma_t| \leq -\mu \sigma_n; \quad \dot{g}_t = |\dot{g}_t| \frac{\sigma_t}{|\sigma_t|}, \quad \sigma_t = \sigma \cdot t \]

Contact boundary conditions \sim \text{unknown Neumann (force) boundary conditions depending on the geometry.}
Account of contact
Coulomb’s friction

- Coulomb’s friction conditions

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- Contact boundary conditions \(\sim\) unknown Neumann (force) boundary conditions depending on the geometry.
Contact detection

- Two FE meshes penetrate each other
  - What penetrates?
    - Nodes, lines, elements?
  - Into what it penetrates?
    - Into elements, under surface?
  - How do we detect such penetration?
Contact detection

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Zoom on penetration
Contact detection

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Volume intersection
Contact detection

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Contact discretization

What is elementary contact contributor?

- Local contacting unit?
- Node-to-node
- Node-to-segment/Node-to-edge/Node-to-surface
- Gauss point to surface

Node-to-node discretization: small deformation/small sliding
Contact discretization

What is elementary contact contributor?

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Node-to-node discretization:
small deformation/small sliding

Large deformation/large sliding
Contact discretization

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Node-to-node discretization: small deformation/small sliding

Node-to-segment discretization: large deformation/large sliding
What is elementary contact contributor?

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Node-to-node discretization: small deformation/small sliding

Contact domain method: large deformation/large sliding
Contact discretization

What is elementary contact contributor?

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- Gauss point to surface

Conception of the contact element

Node-to-node discretization: small deformation/small sliding

Contact domain method: large deformation/large sliding
Plan

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4. Contact discretization methods
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Spatial search and local detection

**Spatial search**
Detection of contacting solids: multibody systems
- Discrete Element Method
- Molecular dynamics

Example of DEM application
[Williams, O’Connor, 1999]

**Local contact detection**
Detection of contacting nodes and surfaces of two discretized solids
- Finite Element Method
- Smoothed Particle Hydrodynamics

Torus–cylinder impact problem
[B. Yang, T.A. Laursen, 2006]
Spatial search and local detection

**Spatial search**
Detection of contacting solids: multibody systems
- Discrete Element Method
- Molecular dynamics

3rd body layer modeling
[V.-D. Nguyen, J. Fortin et al., 2009]

**Local contact detection**
Detection of contacting nodes and surfaces of two discretized solids
- Finite Element Method
- Smoothed Particle Hydrodynamics

Buckling with self-contact
[T. Belytschko, W.K. Liu, B. Moran, 2000]
Local contact detection

Basic ideas

- How to detect contact?
- What penetrates and where?
- What/where approach
Local contact detection

Basic ideas

- How to detect contact?
- What penetrates and where?
- What/where approach, for example,
  - What? Slave nodes
  - Where? Under master surface
Local contact detection

Basic ideas

- How to detect contact?
- **What** penetrates and **where**?
- **What/where** approach, for example,
  - What? **Slave** nodes
  - Where? Under **master** surface
- Assymetry of contacting surfaces
Local contact detection

Basic ideas

- How to detect contact?
- What penetrates and where?
- What/where approach, for example,
  - What? **Slave** nodes
  - Where? Under **master** surface
- Assymetry of contacting surfaces

**New paradigm**

**BUT!** We need to detect contact before any penetration occurs!
Local contact detection

Basic ideas

- How to detect contact?
- What penetrates and where?
- What/where approach, for example,
  - What? **Slave** nodes
  - Where? Under **master** surface
- Assymetry of contacting surfaces

**New paradigm**

**BUT!** We need to detect contact before any penetration occurs!

**Contact detection idea**

In general the detection consists in checking if **slave nodes** are close enough to the **master surface**
**Local contact detection**

**Basic ideas**

- How to detect contact?
- **What** penetrates and **where**?
- **What/where** approach, for example,
  - What? **Slave** nodes
  - Where? Under **master** surface
- Assymetry of contacting surfaces

---

**New paradigm**

**BUT!** We need to detect contact before any penetration occurs!

---

**Contact detection idea**

In general the detection consists in checking if **slave nodes** are close enough\(^a\) to the **master surface**

\(^a\)What does it mean close enough?
Maximal detection distance concept $d_{\text{max}}$

If a slave node is closer than $d_{\text{max}}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

Solid with slave nodes and solid with a master surface.
Maximal detection distance concept $d_{\text{max}}$

If a slave node is closer than $d_{\text{max}}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

External detection zone dist $< d_{\text{max}}$ and $g_n \geq 0$
Maximal detection distance concept \( d_{\text{max}} \)

If a slave node is closer than \( d_{\text{max}} \) to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

Internal detection zone dist < \( d_{\text{max}} \) and \( g_n < 0 \)
Maximal detection distance concept $d_{\text{max}}$

If a slave node is closer than $d_{\text{max}}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

Contact detection zone dist $< d_{\text{max}}$
Maximal detection distance concept $d_{\text{max}}$

If a **slave node** is closer than $d_{\text{max}}$ to the **master surface**, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

Verification if slave nodes are in the detection zone
Maximal detection distance concept $d_{\text{max}}$

If a slave node is closer than $d_{\text{max}}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

No contact detected!

No slave nodes in the detection zone
Maximal detection distance concept $d_{\text{max}}$

If a **slave node** is closer than $d_{\text{max}}$ to the **master surface**, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

No slave nodes in the detection zone
Maximal detection distance concept $d_{\text{max}}$

If a slave node is closer than $d_{\text{max}}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

Solid with slave nodes and solid with a master surface.
Maximal detection distance concept $d_{\text{max}}$

If a slave node is closer than $d_{\text{max}}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

Contact detection zone dist $< d_{\text{max}}$
Maximal detection distance concept $d_{\text{max}}$

If a slave node is closer than $d_{\text{max}}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

Verification if slave nodes are in the detection zone
Maximal detection distance concept $d_{\text{max}}$

If a slave node is closer than $d_{\text{max}}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

Some slave nodes are in the detection zone. One node is missed!
Maximal detection distance concept \( d_{\text{max}} \)

If a slave node is closer than \( d_{\text{max}} \) to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

Create contact elements with detected slave nodes.
Maximal detection distance concept $d_{\text{max}}$

If a slave node is closer than $d_{\text{max}}$ to the master surface, then this slave node is considered as possibly contacting at current load step and a contact element is to be created.

How to choose $d_{\text{max}}$?

**Dangerous solution**
- Make the user responsible for this choice

**Practical solutions**: choose accordingly to
- master surface mesh;
- boundary conditions (maximal variation of displacement of contacting surface nodes during one iteration/increment)
- use both criterions.
Discretized master surface

- **Master surface** consists of many elementary **master surfaces**.
- The aim is to find for each **slave node** the associated **master surface**.

Triangles - **slave nodes** and circles - **master nodes** which are connected by **master surfaces**.
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**Contact detection zone.**
What does it consist of?
- **Master surface** consists of many elementary **master surfaces**.
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Zoom on the **master contact surface** and the associated detection zone.
Discretized master surface

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Contact detection zone is nothing but the region, where each point is closer to the **master surface** than $d_{\text{max}}$. 
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Each contact element consists of a **slave node** and of the **master surface** onto which it projects, i.e. the closest **master surface**.
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Closest points definition

How to define the closest point $\rho^*$ onto the master surface $\Gamma_m$ for a given slave node $r_s$?
Closest points definition

- How to define the closest point $\rho^*$ onto the master surface $\Gamma_m$ for a given slave node $r_s$?

  - Definition of the closest point $\rho^*$

\[
\rho^* \in \Gamma_m : \forall \rho \in \Gamma_m, |r_s - \rho^*| \leq |r_s - \rho|
\]
Closest points definition

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$$\rho^* \in \Gamma_m : \forall \rho \in \Gamma_m, |r_s - \rho^*| \leq |r_s - \rho|$$

- Functional for parametrized surface $\rho = \rho(\xi)$:

$$F(r_s, \xi) = \frac{1}{2} (r_s - \rho(\xi))^2$$
Closest points definition

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Functional for parametrized surface $\rho = \rho(\xi)$:

$$F(r_s, \xi) = \frac{1}{2} (r_s - \rho(\xi))^2$$

In general case, nonlinear minimization problem:

$$\min_{\xi \in [0;1]} F(r_s, \xi) \Rightarrow \xi^*: \forall \xi \in [0;1], |r_s - \rho(\xi^*)| \leq |r_s - \rho(\xi)|$$

$$\min_{\xi \in [0;1]} F(r_s, \xi) \iff (r_s - \rho(\xi^*)) \cdot \frac{\partial \rho}{\partial \xi}\bigg|_{\xi^*} = 0$$
How to define the closest point $\mathbf{r}^*$ onto the master surface $\Gamma_m$ for a given slave node $\mathbf{r}_s$?

\[
\min_{\xi \in [0;1]} F(\mathbf{r}_s, \xi) \Leftrightarrow (\mathbf{r}_s - \mathbf{r}^*(\xi)) \cdot \left. \frac{\partial \mathbf{r}}{\partial \xi} \right|_{\xi^*} = 0
\]
Existance/unicateness of the closest point

- Does the projection always exist?
Existance/uniqueness of the closest point

- Does the projection always exist? **No!**
Existance/uniqueness of the closest point

- Does the projection always exist? **No!**
  - Only if the master surface is smooth $\rho(\xi) \in C^1(\Gamma_m)$

$$ (r_s - \rho(\xi^*)) \cdot \left. \frac{\partial \rho}{\partial \xi} \right|_{\xi^*} = 0 $$

- In the FEM the surface is not obligatory smooth, only continuous $\rho(\xi) \in C^0(\Gamma_m)$
- Even if a projection exists it can be **non-unique**.
Existance/uniqueness of the closest point

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For more details see **Contact geometry section**
Blind spots

Blind spot problem

- Blind spots – gaps in the detection zone.
- Do not miss slave nodes situated in blind spots.
- Types of blind spots: external, internal, due to symmetry.

Do not miss slave nodes situated in blind spots.

Types of blind spots: external, internal, due to symmetry.
Who is master, who is slave?
Social inequality in contact problems

Master-slave definition

- The choice of master and slave surfaces is not random.
- Incorrect choice leads to meaningless solutions.

Initial FE mesh
Incorrect master-slave choice 😞
Correct master-slave choice 😊
Contact detection in global algorithm

At the beginning of each increment (loading step)

+ fast;
+ good convergence;
+ stable;
- lack of accuracy.

At the beginning of each iteration (convergence step)

- slow;
- infinite looping;
- not stable;
+ more accurate.
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Contact detection techniques

- **All-to-all detection**
  - Direct all-to-all detection
    - Slave nodes to master segments.
  - Indirect all-to-all detection
    - Slave nodes to master nodes.
    - Slave node to attached master segments.

- **Advantages:**
  - Simple implementation.

- **Drawbacks:**
  - Time consuming $O(N^2)$;
  - Blind spots/passing by nodes.

- **Elaborated techniques**
  - Bounding boxes
    - Account only close regions.
  - Regular grid methods: bucket [Benson].
    - Detect only into a cell and in neighbouring cells
  - Sorting methods: heap sort, octree [Williams, O’Connor].
    - Data tree construction and tree search.

- **Advantages:**
  - Relatively fast $O(N)$, $O(N \log N)$.

- **Drawbacks:**
  - Blind spots/passing by nodes.
Contact detection techniques

- **All-to-all detection**

Example: two curved contacting surfaces – 1 million of slave nodes against 1 million of master segments.

- **Elaborated techniques**
Contact detection techniques

- **All-to-all detection**
  
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- **Elaborated techniques**
Contact detection techniques

- All-to-all detection
- Elaborated techniques

Example: two curved contacting surfaces –
1 million of slave nodes against 1 million of master segments.

Indirect all-to-all technique

- 187 hours(!)
Contact detection techniques

- **All-to-all detection**

  Example: two curved contacting surfaces – 1 million of slave nodes against 1 million of master segments.

- **Elaborated techniques**

  - Indirect all-to-all technique
    - 187 hours(!)
  - Bounding box + grid detection
    - 1 minute
Summary
Contact detection

- Global search/local contact detection;
- slave-master or what-where approach;
- conception of the maximal detection distance and its choice;
- contact geometry and contact detection - closest point definition;
- existence and uniqueness of the closest point;
- from continuous and smooth to discretized $C^0$ surface;
- attention - blind spots;
- detect contact at the beginning of increment;
- different detection techniques;
- self-contact detection.

- contact detection is strongly connected with contact geometry and contact discretization.

[B. Yang, T.A. Laursen, 2006]
Plan

1. Introduction
2. Contact detection
3. Contact geometry
4. Contact discretization methods
5. Solution of contact problem
6. Finite Element Analysis of contact problems
7. Numerical examples
Introduction

- Geometry is a foundation for
  - master-slave approach;
  - detection;
  - discretization method;
  - solution.

- Geometrical quantities
  - penetration or normal gap $g_n$ - normal contact;
  - tangential sliding $\Delta g_t$ - frictional effects.
Challenges in contact geometry

Challenges

- requirement of smooth surface.
- non-uniqueness of projection.
Challenges in contact geometry

Challenges

- asymmetry of contacting surfaces;
- requirement of smooth surface;
- non-uniqueness of projection;

\[ a \text{something penetrates into something, something slides over something} \]
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- asymmetry of contacting surfaces;
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Challenges in contact geometry

Challenges

- asymmetry of contacting surfaces;
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*something penetrates into something, something slides over something*
Smoothness of contact surface

Do real surfaces are smooth or not?
- It depends on the scale.
- The smaller scale, the higher roughness.
- Fractal surface.

Surface deforms even without contact
- macro deformation;
- escaping dislocations and twins;
- relaxation at nano-scale.

Finite element surface is not smooth
- convergence problems;
- unphysical oscillations;
- remedy - special smoothing techniques.
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Polished metal surface
400x600 microns specimen

Fractal surface
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Deformation twinnings in coating [Forest, 2000]
Escaping dislocations [Fivel, 2009]
## Smoothness of contact surface

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*Deformation twinnings in coating*  
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- Convergence problems;
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2D surface smoothing with Bezier curves
Smoothness of contact surface

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- macro deformation;
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Finite element surface is not smooth
- convergence problems;
- unphysical oscillations;
- remedy - special smoothing techniques.
Account of geometrical quantities
First order variations

Contact in the weak form

- Geometrical quantities: normal gap $g_n = g_n(x)$ and tangential sliding $\dot{g}_t = \dot{g}_t(x^0, x)$
- Virtual work at contact surface $\delta W^{cont} = \delta W_n + \delta W_t$
  - normal contact $W_n(\sigma_n, g_n) \Rightarrow \delta W_n(\sigma_n, g_n, \delta \sigma_n, \delta g_n)$
  - frictional contact $W_t(\sigma_t, \dot{g}_t) \Rightarrow \delta W_t(\sigma_t, \dot{g}_t, \delta \sigma_t, \delta \dot{g}_t)$
- Resulting nonlinear equation $R \left( \delta W^{int}, \delta W^{ext}, \delta W^{cont} \right) = 0$

Need of analytical expressions

- First order variation of the normal gap and tangential sliding

\[ \delta g_n = \frac{\partial g_n}{\partial x} \cdot \delta x \quad \delta \dot{g}_t = \frac{\partial \dot{g}_t}{\partial x} \cdot \delta x \]
Account of geometrical quantities
Second order variations

Linearization of the weak form

- Resulting nonlinear equation $R \left( \delta W^{\text{int}}, \delta W^{\text{ext}}, \delta W^{\text{cont}} \right) = 0$
- Linearization $R(x) = 0 \Rightarrow R|_{x_0} + \Delta R(x)|_{x_0} \approx 0$
- $\Delta R(x) = \Delta \delta W^{\text{int}} + \Delta \delta W^{\text{ext}} + \Delta \delta W^{\text{cont}}$
  - normal contact $\Delta \delta W_n(\sigma_n, g_n, \delta \sigma_n, \delta g_n, \Delta \delta \sigma_n, \Delta \delta g_n)$
  - frictional contact $\Delta \delta W_t(\sigma_t, \dot{g}_t, \delta \sigma_t, \delta \dot{g}_t, \Delta \delta \sigma_t, \Delta \delta \dot{g}_t)$

Need of analytical expressions

- Second order variation of the normal gap and tangential sliding
  - $\Delta \delta g_n = \Delta x \cdot \frac{\partial^2 g_n}{\partial x^2} \cdot \delta x$
  - $\Delta \delta \dot{g}_t = \Delta x \cdot \frac{\partial^2 \dot{g}_t}{\partial x^2} \cdot \delta x$
Point and surface
Master-slave approach

Master-slave

Slave node penetrates under and slides over the master surface.

- Slave node – point \( r_s \)
- Master surface – \( \rho(\xi) \)
- Surface parametrization \( \xi = \{\xi_1, \xi_2\} \)
- Projection of the slave node \( \rho(\xi_p) \)
- Normal to the master surface \( n(\xi_p) \)

Geometrical quantities

- Normal gap \( g_n = (r_s - \rho) \cdot n \)
- Tangential sliding \( \dot{g}_t dt = \delta \rho(\xi) \)
Point and surface
Master-slave approach

**Master-slave**

**Slave node** penetrates under and slides over the *master surface*.

- **Slave node** – point \( r_s = r_s(t) \)
- **Master surface** – \( \rho(\xi) = \rho(t, \xi) \)
- **Surface parametrization** \( \xi = \{\xi_1, \xi_2\} \)
- **Projection of the slave node**
  \[ \rho(\xi_p) = \rho(t, \xi(r_s(t))) \]
- **Normal to the master surface**
  \[ n(\xi_p) = n(t, \xi(r_s(t))) \]

**Geometrical quantities**

- **Normal gap**
  \[ g_n(t) = (r_s(t) - \rho(t, \xi(r_s(t))) \cdot n(t, \xi(r_s(t)))) \]
- **Tangential sliding**
  \[ \dot{g}_t(t, \xi(r_s(t))) dt = \delta \rho(t, \xi(r_s(t))) \]
Continuum geometrical formulation

Covariant and contravariant bases, fundamental surface tensors

- **Covariant surface basis**
  \[ \frac{\partial \rho}{\partial \xi} = \left\{ \frac{\partial \rho}{\partial \xi_1}, \frac{\partial \rho}{\partial \xi_2} \right\} \]

- **Contravariant surface basis**
  \[ \frac{\partial \hat{\rho}}{\partial \xi} = \left\{ \frac{\partial \rho}{\partial \xi^1}, \frac{\partial \rho}{\partial \xi^2} \right\} \]

- **Basis change**
  \[ \frac{\partial \hat{\rho}}{\partial \xi} = A^{-1} \frac{\partial \rho}{\partial \xi} \frac{\partial \rho}{\partial \xi^i} = a^{ij} \frac{\partial \rho}{\partial \xi^j} \]

- **1st fundamental covariant surface tensor**
  \[ A \sim a_{ij} = \frac{\partial \rho}{\partial \xi_i} \cdot \frac{\partial \rho}{\partial \xi_j} \]

- **1st fundamental contravariant surface tensor**
  \[ A^{-1} \sim a^{ij} = \frac{\partial \hat{\rho}}{\partial \xi^i} \cdot \frac{\partial \hat{\rho}}{\partial \xi^j} \]

- **2nd fundamental surface tensor**
  \[ H \sim h_{ij} = n \cdot \frac{\partial^2 \rho}{\partial \xi_i \partial \xi_j} \]
#### Continuum geometrical formulation

**First order variations**

- First order variation of the normal gap $\delta g_n$

\[
\delta g_n = n \cdot (\delta r_s - \delta \rho)
\]

- First order variation of the surface parameter $\delta \xi$

\[
\delta \xi = (A - g_n H)^{-1} \cdot \left( \frac{\partial \rho}{\partial \xi} \cdot (\delta r_s - \delta \rho) + g_n n \cdot \delta \frac{\partial \rho}{\partial \xi} \right)
\]
Continuum geometrical formulation

Second order variations

- Second order variation of the normal gap $\delta g_n$

\[
\Delta \delta g_n = -\mathbf{n} \cdot \left( \delta \frac{\partial \rho}{\partial \xi} \cdot \Delta \xi + \Delta \frac{\partial \rho}{\partial \xi} \cdot \delta \xi + \delta \xi \cdot \frac{\partial^2 \rho}{\partial \xi^2} \cdot \Delta \xi \right) + \\
+ g_n \delta \xi \cdot \mathbf{H} \cdot \mathbf{A}^{-1} \cdot \mathbf{H} \cdot \Delta \xi + \\
+ g_n \left( \mathbf{n} \cdot \delta \frac{\partial \rho}{\partial \xi} \right) \cdot \mathbf{A}^{-1} \left( \Delta \frac{\partial \rho}{\partial \xi} \cdot \mathbf{n} \right)
\]

- Second order variation of the surface parameter $\delta \xi$

\[
\Delta \delta \xi = (g_n \mathbf{H} - \mathbf{A})^{-1} \cdot \left[ \delta \frac{\partial \rho}{\partial \xi} \cdot \left( \delta \frac{\partial \rho}{\partial \xi} \cdot \Delta \xi + \Delta \frac{\partial \rho}{\partial \xi} \cdot \delta \xi + \delta \xi \cdot \frac{\partial^2 \rho}{\partial \xi^2} \cdot \Delta \xi \right) \right] \\
- g_n \mathbf{n} \cdot \left( \delta \frac{\partial^2 \rho}{\partial \xi^2} \cdot \Delta \xi + \Delta \frac{\partial^2 \rho}{\partial \xi^2} \cdot \delta \xi + \delta \xi \cdot \frac{\partial^3 \rho}{\partial \xi^3} \cdot \Delta \xi \right) \\
+ \left\{ \Delta \frac{\partial \rho}{\partial \xi} \cdot \mathbf{n} + \mathbf{H} \cdot \Delta \xi \right\} \cdot \left( \mathbf{l} \left\{ \mathbf{n} \cdot (\delta \rho - \delta \mathbf{r}_s) \right\} + g_n \mathbf{A}^{-1} \cdot \frac{\partial \rho}{\partial \xi} \cdot \left\{ \delta \frac{\partial \rho}{\partial \xi} + \frac{\partial^2 \rho}{\partial \xi^2} \cdot \delta \xi \right\} \right) \\
+ \left\{ \delta \frac{\partial \rho}{\partial \xi} \cdot \mathbf{n} + \mathbf{H} \cdot \delta \xi \right\} \cdot \left( \mathbf{l} \left\{ \mathbf{n} \cdot (\Delta \rho - \Delta \mathbf{r}_s) \right\} + g_n \mathbf{A}^{-1} \cdot \frac{\partial \rho}{\partial \xi} \cdot \left\{ \Delta \frac{\partial \rho}{\partial \xi} + \frac{\partial^2 \rho}{\partial \xi^2} \cdot \Delta \xi \right\} \right)
\]

(1)

(2)
Approximation of zero penetration

Normal gap is small $g_n \approx 0$. 

\[ \delta g_n = \mathbf{n} \cdot (\delta r_s - \delta \rho) \]
\[ \delta \xi = A^{-1} \cdot \frac{\partial \rho}{\partial \xi} \cdot (\delta r_s - \delta \rho) \]

\[ \Delta \delta g_n = -\mathbf{n} \cdot \left( \delta \frac{\partial \rho}{\partial \xi} \cdot \Delta \xi + \Delta \frac{\partial \rho}{\partial \xi} \cdot \delta \xi + \delta \xi \cdot \frac{\partial^2 \rho}{\partial \xi^2} \cdot \Delta \xi \right) \]

\[ \Delta \delta \xi = -A^{-1} \cdot \left[ \frac{\partial \rho}{\partial \xi} \cdot \left( \delta \frac{\partial \rho}{\partial \xi} \cdot \Delta \xi + \Delta \frac{\partial \rho}{\partial \xi} \cdot \delta \xi + \delta \xi \cdot \frac{\partial^2 \rho}{\partial \xi^2} \cdot \Delta \xi \right) + \right. \]
\[ + \left\{ \Delta \frac{\partial \rho}{\partial \xi} \cdot \mathbf{n} + H \cdot \Delta \xi \right\} \cdot (I \{\mathbf{n} \cdot (\Delta \rho - \Delta r_s)\}) + \]
\[ + \left\{ \delta \frac{\partial \rho}{\partial \xi} \cdot \mathbf{n} + H \cdot \delta \xi \right\} \cdot (I \{\mathbf{n} \cdot (\Delta \rho - \Delta r_s)\}) \]
Discretized contact geometry
From continuum formulation to the Finite Element Method

Finite element method formalism

\[
\mathbf{r} = \sum_{i=1}^{N} \phi_i(\xi) \mathbf{x}_i = \sum_{i=1}^{N} \phi_i(\xi_1, \xi_2) \mathbf{x}_i
\]

\[
[\mathbf{X}] = [\mathbf{X}(t)] = [\mathbf{x}_0(t), \mathbf{x}_1(t), \ldots, \mathbf{x}_N(t)]^T;
\]

\[
[\Phi] = [\Phi(\xi)] = [0, \phi_1(\xi), \ldots, \phi_N(\xi)]^T;
\]

\[
[\Phi'] = \left[ \frac{\partial \Phi(\xi)}{\partial \xi_i} \right] = [0, \phi_{1,i}, \ldots, \phi_{N,i}]^T;
\]

\[
\mathbf{r}_s = \mathbf{r}_s(t) = \mathbf{x}_0(t) = \left[ \mathbf{S}_0 \right]^T [\mathbf{X}], \text{ where } [\mathbf{S}_0] = [1, 0, \ldots, 0]^T.
\]

\[
\rho = \rho(t, \xi_p) = \phi_i(\xi_p) \mathbf{x}_i = [\Phi(\xi_p)]^T [\mathbf{X}(t)], = [\Phi]^T [\mathbf{X}]
\]

\[
\rho_i = \frac{\partial \rho}{\partial \xi_i} = \left. \frac{\partial \rho(t, \xi)}{\partial \xi_i} \right|_{\xi_p} = \left[ \frac{\partial \Phi(\xi)}{\partial \xi_i} \right]_{\xi_p}^T [\mathbf{X}(t)] = [\Phi']^T [\mathbf{X}]
\]
Discretized contact geometry
From continuum formulation to the Finite Element Method II

First variations of geometrical quantities

\[
\delta g_n = \begin{bmatrix}
  n \\
  -\phi_1 n \\
  \vdots \\
  -\phi_N n
\end{bmatrix}^T \begin{bmatrix}
  \delta x_0 \\
  \delta x_1 \\
  \vdots \\
  \delta x_N
\end{bmatrix} = [\nabla g_n]^T \cdot \delta [X]
\]  

(4)

\[
\delta \xi_i = c_{ij} \begin{bmatrix}
  \frac{\partial \rho}{\partial \xi_j} \\
  -\frac{\partial \rho}{\partial \xi_j} \phi_1 + g_n n \phi_{1,j} \\
  \vdots \\
  -\frac{\partial \rho}{\partial \xi_j} \phi_N + g_n n \phi_{N,j}
\end{bmatrix}^T \begin{bmatrix}
  \delta x_0 \\
  \delta x_1 \\
  \vdots \\
  \delta x_N
\end{bmatrix} = [\nabla \xi_i]^T \cdot \delta [X]
\]  

(5)
Second order variation of the normal gap $\Delta \delta g_n$

$$\Delta \delta g_n = \delta [X]^T \cdot \left\{ -\mathbf{n} \left[ \Phi'_i \right] \otimes [\nabla \xi_i]^T - [\nabla \xi_i] \otimes \left[ \Phi'_j \right]^T \mathbf{n} \\
- \left( h_{ij} - g_n h_{ik} a^{km} h_{mj} \right) [\nabla \xi_i] \otimes [\nabla \xi_j]^T \\
+ g_n a^{ij} \mathbf{n} \left[ \Phi'_i \right] \otimes \left[ \Phi'_j \right]^T \mathbf{n} \right\} \cdot \Delta [X] =$$

$$= \delta [X]^T \cdot [\nabla \nabla g_n] \cdot \Delta [X] =$$

(6)
Discretized contact geometry
From continuum formulation to the Finite Element Method II

Second order variation of surface parameter $\Delta \delta \xi$

\[
\Delta \delta \xi_i = \delta [X]^T \cdot \left\{ -\varepsilon_{ij} \left[ \frac{\partial \rho}{\partial \xi_j} \otimes [\nabla \xi_k]^T + [\nabla \xi_k] \otimes \frac{\partial \rho}{\partial \xi_j} \right] \phi'_k \right. \\
+ \left. \left( \frac{\partial \rho}{\partial \xi_j} \cdot \frac{\partial^2 \rho}{\partial \xi_k \partial \xi_m} + g_n \cdot \frac{\partial^3 \rho}{\partial \xi_k \partial \xi_j \partial \xi_m} \right) [\nabla \xi_k] \otimes [\nabla \xi_m]^T \right. \\
+ \left. g_n \left[ \phi''_{jk} \right] n \otimes [\nabla \xi_k]^T + g_n [\nabla \xi_k] \otimes n \left[ \phi''_{jk} \right]^T \right. \\
- \left. \delta_{kj} \left( [\nabla g_n] \otimes \left( n \left[ \phi'_k \right]^T + h_{ks} [\nabla \xi_s]^T \right) + \left( [\phi'_k] n + h_{ks} [\nabla \xi_s] \right) \otimes [\nabla g_n] \right) \right. \\
+ \left. g_{na}^k \left( \frac{\partial \rho}{\partial \xi_m} \otimes \left( n \left[ \phi'_k \right]^T + h_{ks} [\nabla \xi_s]^T \right) + \left( [\phi'_k] + h_{ks} [\nabla \xi_s] \right) \otimes \frac{\partial \rho}{\partial \xi_m} \left[ \phi'_j \right]^T \right) \right. \\
+ \left. g_{na}^k \left( \frac{\partial \rho}{\partial \xi_m} \cdot \frac{\partial^2 \rho}{\partial \xi_j \partial \xi_l} \right) \left( [\nabla \xi_l] \otimes \left( n \left[ \phi'_k \right]^T + h_{ks} [\nabla \xi_s]^T \right) + \left( n \left[ \phi'_k \right] + h_{ks} [\nabla \xi_s] \right) \otimes [\nabla \xi_l]^T \right) \right. \\
+ \left. + \left( n \left[ \phi'_k \right] + h_{ks} [\nabla \xi_s] \right) \otimes [\nabla \xi_l]^T \right) \} \cdot \Delta [X] = \\
= \delta [X]^T \cdot [\nabla \xi_i] \cdot \Delta [X]
\]
Plan

1. Introduction
2. Contact detection
3. Contact geometry
4. Contact discretization methods
5. Solution of contact problem
6. Finite Element Analysis of contact problems
7. Numerical examples
Discretization of the contact area into elementary units responsible for the contact stress transmission from one contacting surface to another.

Units:
- surface nodes;
- surface of elements;
- Gauss points onto surfaces;

Problem types:
- small deformation - no slip;
- large deformation - arbitrary slip.

Discretization method and assymetry:
- methods which enforce assymetry of surfaces:
  - node-to-segment.
- methods which reduce this assymetry:
  - segment-to-segment, mortar, Nitsche;
  - contact domain method.
NTN – Node-to-node

Node-to-node discretization

[Francavilla & Zienkiewicz, 1975], [Oden, 1981], [Kikuchi & Oden, 1988]

Advantages:

😊 very simple;
😊 passes Taylor’s test\(^1\).

\(^1\) Taylor’s patch test requires that a uniform contact stress transmittes correctly from one contacting surface to another. See section Finite Element Analysis.

Scheme of two conforming meshes. Pairing nodes form NTN contact elements.
Node-to-node discretization

[Francavilla & Zienkiewicz, 1975], [Oden, 1981], [Kikuchi & Oden, 1988]

**Advantages:**

👍 very simple;
👍 passes Taylor’s test\(^1\).

**Drawbacks:**

😊 small deformation;
😊 small slip;
😊 requires conforming FE meshes.

Scheme of two conforming meshes. Pairing nodes form NTN contact elements.

---

\(^1\) Taylor’s patch test requires that a uniform contact stress transmistes correctly from one contacting surface to another. See section **Finite Element Analysis**.
Definition of the normal for the node-to-node discretization

Two FE mesh with matching nodes in the contact zone
NTN – Node-to-node

Definition of the normal for the node-to-node discretization

NTN contact element detection
Definition of the normal for the node-to-node discretization

Master-slave discretization
NTN – Node-to-node

Definition of the normal for the node-to-node discretization

Normals on the master surface
Definition of the normal for the node-to-node discretization

Definition of the normals at master nodes as an average of the normals of adjacent segments
NTN – Node-to-node

Definition of the normal for the node-to-node discretization

Definition of the normals at master nodes
Node-to-segment discretization

[Hughes, 1977] [Hallquist, 1979] [Bathe & Chaudhary, 1985] [Wriggers et al., 1990]

Advantages:

😊 simple;
😊 large deformations and slip;
😊 mesh independent.

Scheme of two non-matching meshes
Node-to-segment discretization

[Hughes, 1977] [Hallquist, 1979] [Bathe & Chaudhary, 1985] [Wriggers et al., 1990]

Advantages:

😊 simple;
😊 large deformations and slip;
😊 mesh independent.

Drawbacks:

😊 does not pass¹ Taylor’s test.

¹[G. Zavarise, L. De Lorenzis, 2009]
A modified NTS algorithm passing the contact patch test
Node-to-segment discretization

Advantages:

😊 simple;
😊 large deformations and slip;
😊 mesh independent.

Drawbacks:

😊 does not pass$^1$ Taylor’s test.

$^1$[G. Zavarise, L. De Lorenzis, 2009]
A modified NTS algorithm passing the contact patch test
Node-to-segment discretization

Advantages:

😊 simple;
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Drawbacks:

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A modified NTS algorithm passing the contact patch test
Node-to-segment discretization

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**Advantages:**

😊 simple;

😊 large deformations and slip;

😊 mesh independent.

**Drawbacks:**

😊 does not pass¹ Taylor’s test.

Remark: for a stable large deformation implementation NTS should be supplemented with Node-to-Vertex and Node-to-Edge.
Segment-to-segment discretization

[Simo et al., 1985], [Zavarise & Wriggers, 1998]

Advantages:

😊 avoids some spurious modes of NTS;
😊 use of higher order shape functions;
😊 large deformations and slip;
😊 mesh independent.

Scheme of two non-matching meshes
Segment-to-segment discretization

**Advantages:**

😊 avoids some spurious modes of NTS;
😊 use of higher order shape functions;
😊 large deformations and slip;
😊 mesh independent.

**Drawbacks:**

😊 complicated segment definition;
😊 only 2D version;
😊 constant contact pressure within one segment.

Scheme of two non-matching meshes
Segment-to-segment discretization

Advantages:

😊 avoids some spurious modes of NTS;
😊 use of higher order shape functions;
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Drawbacks:

😊 complicated segment definition;
😊 only 2D version;
😊 constant contact pressure within one segment.

[Simó et al., 1985], [Zavarise & Wriggers, 1998]

Projections of both surfaces
Segment-to-segment discretization

Advantages:

😊 avoids some spurious modes of NTS;
😊 use of higher order shape functions;
😊 large deformations and slip;
😊 mesh independent.

Drawbacks:

😢 complicated segment definition;
😢 only 2D version;
😢 constant contact pressure within one segment.

[Simó et al., 1985], [Zavarise & Wriggers, 1998]
Segment-to-segment discretization

[Simo et al., 1985], [Zavarise & Wriggers, 1998]

Advantages:

😊 avoids some spurious modes of NTS;
😊 use of higher order shape functions;
😊 large deformations and slip;
😊 mesh independent.

Drawbacks:

😊 complicated segment definition;
😊 only 2D version;
😊 constant contact pressure within one segment.
Mortar and Nitsche discretizations


Advantages:

- passes Taylor’s test;
- correct contact stress distribution within contact element;
- use of any order shape functions;
- large deformations and slip;
- mesh independent.
Mortar and Nitsche discretizations


Advantages:

😊 passes Taylor’s test;
😊 correct contact stress distribution within contact element;
😊 use of any order shape functions;
😊 large deformations and slip;
😊 mesh independent.

Drawbacks:

😊 very complicated implementation\(^1\);
😊 stability problem for curved surfaces.

\(^1\)“3D implementation is a nightmare, but it’s feasible.”
T.A. Laursen about mortar method, ECCM, 2010
Contact domain method for discretization

[Oliver, Hartmann et al. 2009]

**Advantages:**

- 😊 passes Taylor’s test;
- 😊 continuous formulation of contact elements;
- 😊 large deformations and slip.

Scheme of two non-matching meshes
CDDM – Contact Domain Method

Contact domain method for discretization

[Oliver, Hartmann et al. 2009]

**Advantages:**

😊 passes Taylor’s test;

😊 continuous formulation of contact elements;

😊 large deformations and slip.

**Drawbacks:**

😊 requirements on the FE mesh in 3D;

😊 not elaborated.

Scheme of two non-matching meshes

1Triangulation problems for arbitrary contacting surfaces in 3D.
Contact domain method for discretization

[Oliver, Hartmann et al. 2009]

Advantages:

😊 passes Taylor’s test;
😊 continuous formulation of contact elements;
😊 large deformations and slip.

Drawbacks:

😊 requirements on the FE mesh in 3D\(^1\);
😊 not elaborated.

\(^1\)Triangulation problems for arbitrary contacting surfaces in 3D.
Contact domain method for discretization

[Oliver, Hartmann et al. 2009]

**Advantages:**

- 😊 passes Taylor’s test;
- 😊 continuous formulation of contact elements;
- 😊 large deformations and slip.

**Drawbacks:**

- 😞 requirements on the FE mesh in 3D;
- 😞 not elaborated.

---

1 Triangulation problems for arbitrary contacting surfaces in 3D.
Smoothing technique

Smoothing of the master surface with
- Hermite polynomials;
- P-splines;
- Bézier curves;
- etc.

Consequences
- fulfills requirements of $C^1$-smoothness all along the master surface;
- nonphysical edge effects;
- complicated in 3D – requires special FE discretizations.
Smoothing technique

Smoothing of the master surface with

- Hermite polynomials;
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Examples of NTS contact elements smoothed with Bézier curves

Scheme of two non-matching meshes
Smoothing technique

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Smoothing of the master surface
Smoothing technique

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Examples of NTS contact elements smoothed with Bézier curves

Contact detection
**Smoothing technique**

**Smoothing of the master surface with**
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- P-splines;
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- etc.

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- fulfills requirements of $C^1$-smoothness all along the master surface;
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- complicated in 3D – requires special FE discretizations.

**Examples of NTS contact elements smoothed with Bézier curves**

Contact element construction (edge contact element)
Smoothing technique

Smoothing of the master surface with
- Hermite polynomials;
- P-splines;
- Bézier curves;
- etc.

Consequences
- fulfills requirements of $C^1$-smoothness all along the master surface;
- nonphysical edge effects;
- complicated in 3D – requires special FE discretizations.

Examples of NTS contact elements smoothed with Bézier curves

Constructed smoothed contact elements
Smoothing technique

Smoothing of the master surface with
- Hermite polynomials;
- P-splines;
- Bézier curves;
- etc.

Consequences
- fulfills requirements of $C^1$-smoothness all along the master surface;
- nonphysical edge effects;
- complicated in 3D – requires special FE discretizations.

Examples of NTS contact elements smoothed with Bézier curves

Ordinary (top) and smoothed (bottom) NTS contact element in 2D
Smoothing technique

Smoothing of the master surface with
- Hermite polynomials;
- P-splines;
- Bézier curves;
- etc.

Consequences
- fulfills requirements of $C^1$-smoothness all along the master surface;
- nonphysical edge effects;
- complicated in 3D – requires special FE discretizations.

Examples of NTS contact elements smoothed with Bézier curves

Ordinary (top) and smoothed (bottom) NTS contact element in 3D
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Introduction

Boundary value problem with constraints

**Continous formulation of boundary value problem**

- **Partial differential equation**
  \[ \nabla \cdot \sigma + f_v = 0 \text{ in } \Omega^{1,2} \]

- **Neumann and Dirichlet boundary conditions**
  \[ \sigma \cdot n = f_0 \text{ at } \Gamma_N^{1,2}, \quad u = u_0 \text{ at } \Gamma_D^{1,2} \]

- **Contact constraints: non-penetration and non-adhesion at } \Gamma_c \text{ — Signorini’s conditions**}
  \[ g_n \sigma_n = 0, \; g_n \geq 0, \; \sigma \leq 0, \; \sigma_n = \sigma \cdot n \]

- **Contact constraints: Coulomb’s friction at } \Gamma_c**
  \[ |\dot{g}_t| (|\sigma_t| + \mu \sigma_n) = 0; \; |\sigma_t| \leq -\mu \sigma_n; \; \dot{g}_t = |\dot{g}_t| \frac{\sigma_t}{|\sigma_t|}, \; \sigma_t = \sigma \cdot t \]
### Continuous formulation of boundary value problem

- **Partial differential equation**
  \[
  \nabla \cdot \sigma + f_v = 0 \quad \text{in } \Omega^{1,2}
  \]

- **Neumann and Dirichlet boundary conditions**
  \[
  \sigma \cdot n = f_0 \quad \text{at } \Gamma^{1,2}_N, \quad u = u_0 \quad \text{at } \Gamma^{1,2}_D
  \]

- **Contact constraints:** non-penetration and non-adhesion at \( \Gamma_c \) - Signorini's conditions
  \[
  g_n \sigma_n = 0, \quad g_n \geq 0, \quad \sigma \leq 0, \quad \sigma_n = \sigma \cdot n
  \]

- **Contact constraints:** Coulomb's friction at \( \Gamma_c \)
  \[
  |\dot{g}_t| \left( |\sigma_t| + \mu \sigma_n \right) = 0; \quad |\sigma_t| \leq -\mu \sigma_n; \quad \dot{g}_t = \frac{\sigma_t}{|\sigma_t|}, \quad \sigma_t = \sigma \cdot t
  \]
Continuous formulation of boundary value problem

- Partial differential equation
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Boundary value problem with constraints

Continuous formulation of boundary value problem

- Partial differential equation
\[ \nabla \cdot \sigma + f_v = 0 \quad \text{in} \quad \Omega^{1,2} \]

- Neumann and Dirichlet boundary conditions
\[ \sigma \cdot n = f_0 \quad \text{at} \quad \Gamma^1_{N}, \quad u = u_0 \quad \text{at} \quad \Gamma^1_D \]

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\[ g_n \sigma_n = 0, \quad g_n \geq 0, \quad \sigma \leq 0, \quad \sigma_n = \sigma \cdot n \]

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Boundary value problem with constraints

Continuous formulation of boundary value problem

- Partial differential equation
\[ \nabla \cdot \sigma + f_v = 0 \text{ in } \Omega^{1,2} \]

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\[ \sigma \cdot n = f_0 \text{ at } \Gamma^{1,2}_N, \quad u = u_0 \text{ at } \Gamma^{1,2}_D \]

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Weak form with contact terms

**Continous formulation of the weak form for contact problems**

- **Weak form**
  \[
  \int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta u d\Omega - \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u d\Gamma = 0
  \]

- **Contact term in the weak form, contact pressure \(\sigma_n^c = \sigma \cdot n\) at \(\Gamma_c\)**
  \[
  \int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta u d\Omega - \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u d\Gamma - \int_{\Gamma_c} \sigma_n^c \delta g_n d\Gamma = 0
  \]

- **Variational inequality \((\sigma_n^c \delta g_n \geq 0)\)**
  \[
  \int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon d\Omega \geq \int_{\Omega^{1,2}} f_v \cdot \delta u d\Omega + \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u d\Gamma
  \]

- **Variational equality**
  \[
  \int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta u d\Omega - \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u d\Gamma + C = 0
  \]
Introduction
Weak form with contact terms

Continous formulation of the weak form for contact problems

- Weak form

\[
\int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon \, d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta u \, d\Omega - \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u \, d\Gamma = 0
\]

- Contact term in the weak form, contact pressure \( \sigma_n^c = \sigma \cdot n \) at \( \Gamma_c \)

\[
\int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon \, d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta u \, d\Omega - \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u \, d\Gamma - \int_{\Gamma_c} \sigma_n^c \delta g_n \, d\Gamma = 0
\]

- Variational inequality \( (\sigma_n^c \delta g_n \geq 0) \)

\[
\int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon \, d\Omega \geq \int_{\Omega^{1,2}} f_v \cdot \delta u \, d\Omega + \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u \, d\Gamma
\]

- Variational equality

\[
\int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon \, d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta u \, d\Omega - \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u \, d\Gamma + C = 0
\]
**Introduction**

Weak form with contact terms

---

**Continuous formulation of the weak form for contact problems**

- **Weak form**

\[
\int_{\Omega^{1,2}} \mathbf{\sigma} \cdot \delta \mathbf{\epsilon} d\Omega - \int_{\Gamma^{1,2}_N} f_0 \cdot \delta \mathbf{u} d\Gamma = 0
\]

- **Contact term in the weak form, contact pressure** \( \mathbf{\sigma}_n^c = \mathbf{\sigma} \cdot \mathbf{n} \) at \( \Gamma_c \)

\[
\int_{\Omega^{1,2}} \mathbf{\sigma} \cdot \delta \mathbf{\epsilon} d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta \mathbf{u} d\Omega - \int_{\Gamma^{1,2}_N} f_0 \cdot \delta \mathbf{u} d\Gamma - \int_{\Gamma_c} \mathbf{\sigma}_n^c \delta g_n d\Gamma = 0
\]

- **Variational inequality** \( (\mathbf{\sigma}_n^c \delta g_n \geq 0) \)

\[
\int_{\Omega^{1,2}} \mathbf{\sigma} \cdot \delta \mathbf{\epsilon} d\Omega \geq \int_{\Omega^{1,2}} f_v \cdot \delta \mathbf{u} d\Omega + \int_{\Gamma^{1,2}_N} f_0 \cdot \delta \mathbf{u} d\Gamma
\]

- **Variational equality**

\[
\int_{\Omega^{1,2}} \mathbf{\sigma} \cdot \delta \mathbf{\epsilon} d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta \mathbf{u} d\Omega - \int_{\Gamma^{1,2}_N} f_0 \cdot \delta \mathbf{u} d\Gamma + C = 0
\]
Weak form with contact terms

Continous formulation of the weak form for contact problems

- **Weak form**

\[
\int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon \, d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta u \, d\Omega - \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u \, d\Gamma = 0
\]

- **Contact term in the weak form, contact pressure** \(\sigma_n^c = \sigma \cdot n\) at \(\Gamma_c\)

\[
\int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon \, d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta u \, d\Omega - \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u \, d\Gamma - \int_{\Gamma_c} \sigma_n^c \delta g_n \, d\Gamma = 0
\]

- **Variational inequality** \((\sigma_n^c \delta g_n \geq 0)\)

\[
\int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon \, d\Omega \geq \int_{\Omega^{1,2}} f_v \cdot \delta u \, d\Omega + \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u \, d\Gamma
\]

- **Variational equality**

\[
\int_{\Omega^{1,2}} \sigma \cdot \delta \varepsilon \, d\Omega - \int_{\Omega^{1,2}} f_v \cdot \delta u \, d\Omega - \int_{\Gamma_{N}^{1,2}} f_0 \cdot \delta u \, d\Gamma + C = 0
\]
Methods for contact resolution

- Variational inequality
  [Duvaut & Lions, 1976], [Kikuchi & Oden, 1988]

- Variational equality\(^1\)
  - optimization methods
  - mathematical programming methods
    [Conry & Siereg, 1971], [Klarbring, 1986]

\(^1\)Often used with so-called active set strategy, which determines which contact elements are active (in contact) and which are not.
Optimization methods

Function to minimize $f(x)$ and constraint $g_i(x) \geq 0$, $i = 1, N$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method
Optimization methods

Function to minimize $f(x)$ and constraint $g_i(x) \geq 0$, $i = 1, N$

- **Penalty method**

$$f_p(x) = f(x) + r \langle g(x) \rangle^2$$

$$\nabla f_p(\bar{x}) = \nabla f(\bar{x}) + 2r \nabla \langle g(\bar{x}) \rangle \nabla g(\bar{x}) = 0$$

$$\bar{x} \xrightarrow{r \rightarrow \infty} x^*$$

- **Lagrange multipliers method**

- **Augmented Lagrangian method**
Optimization methods

Function to minimize $f(x)$ and constraint $g_i(x) \geq 0, \ i = 1, N$

- **Penalty method**
- **Lagrange multipliers method**

$$\mathcal{L}(x, \lambda) = f(x) + \lambda_i g_i(x)$$

$$\min_{g(x) \geq 0} \{f(x)\} \quad \xymatrix{ \x^* = \bar{x} \ar @{<}[r] & \min \{\mathcal{L}(x, \lambda)\}}$$

$$\nabla_{x,\lambda} \mathcal{L} = \begin{bmatrix} \nabla_x f(x) + \lambda_i \nabla_x g_i(x) \\ g_i(x) \end{bmatrix} = 0, \ \lambda_i \leq 0$$

- **Augmented Lagrangian method**
Optimization methods

Function to minimize $f(x)$ and constraint $g_i(x) \geq 0, \ i = 1, N$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

$$
\mathcal{L}_a(x, \lambda) = f(x) + \lambda_i g_i(x) + r \langle g_i(x) \rangle^2
$$

$$
\min_{g(x) \geq 0} \{f(x)\} \rightarrow x^* = \bar{x} \leftarrow \min \{\mathcal{L}_a(x, \lambda)\}
$$

$$
\nabla_{x, \lambda} \mathcal{L}_a = \begin{bmatrix}
\nabla_x f(x) + \lambda_i \nabla_x g_i(x) + 2r \nabla \langle g_i(x) \rangle \nabla g_i(x)
\end{bmatrix}_{g_i(x)} = 0
$$
Optimization methods

Function: $f(x) = x^2 + 2x + 1$

Constrain: $g(x) = x \geq 0$

Solution: $x^* = 0$
Function: $f(x) = x^2 + 2x + 1$

Constrain: $g(x) = x \geq 0$

Solution: $x^* = 0$
Optimization methods
Demonstration :: penalty method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- Penalty method

\[ f_p(x) = f(x) + r \langle -g(x) \rangle^2 \]
Optimization methods
Demonstration :: penalty method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- **Penalty method**

\[ f_p(x) = f(x) + r \langle -g(x) \rangle^2 \]

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]
Optimization methods
Demonstration :: penalty method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- Penalty method

\[ f_p(x) = f(x) + r \langle -g(x) \rangle^2 \]

For the function:

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0 \]

The solution is:

\[ x^* = 0 \]
Optimization methods

Demonstration :: penalty method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- **Penalty method**

\[ f_p(x) = f(x) + r \langle -g(x) \rangle^2 \]

\[ f(x) \]

\[ r = 10 \]

\[ r = 1 \]

\[ f(x) \]

\[ r = 10 \]

\[ f(x) \]

\[ r = 1 \]

\[ f(x) \]

\[ r = 10 \]

\[ f(x) \]

\[ r = 1 \]

\[ f(x) \]
Optimization methods
Demonstration :: penalty method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

**Penalty method**

\[ f_p(x) = f(x) + r \langle -g(x) \rangle^2 \]

\[ r = 50 \]
Optimization methods
Demonstration :: penalty method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- **Penalty method**

\[ f_p(x) = f(x) + r \langle -g(x) \rangle^2 \]

**Advantages 😊**
- simple physical interpretation;
- no additional degrees of freedom;
- smooth functional.

**Drawbacks 😞**
- solution is not exact:
  - too small penalty $\rightarrow$ large penetration;
  - too large penalty $\rightarrow$ ill-conditioning of the global matrix;
- user has to choose penalty $r$ properly.
Optimization methods
Demonstration :: Lagrange multipliers method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- **Lagrange multipliers method**

\[ \mathcal{L}(x, \lambda) = f(x) + \lambda g(x) \rightarrow \text{Saddle point} \rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda) \]
Optimization methods
Demonstration :: Lagrange multipliers method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- **Lagrange multipliers method**

\[ \mathcal{L}(x, \lambda) = f(x) + \lambda g(x) \rightarrow \text{Saddle point} \rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda) \]

Additional unknown \( \lambda \)
Optimization methods

Demonstration :: Lagrange multipliers method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

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Optimization methods

Demonstration :: Lagrange multipliers method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

■ Lagrange multipliers method

\[ \mathcal{L}(x, \lambda) = f(x) + \lambda g(x) \rightarrow \text{Saddle point} \rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda) \]
Optimization methods
Demonstration :: Lagrange multipliers method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- Lagrange multipliers method

\[ \mathcal{L}(x, \lambda) = f(x) + \lambda g(x) \rightarrow \text{Saddle point} \rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda) \]

Advantages 😊
- exact solution.

Drawbacks 😞
- Lagrangian is not smooth;
- additional degrees of freedom increase the problem.
Optimization methods

Demonstration :: Augmented Lagrangian method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

Augmented Lagrangian method

\[ \mathcal{L}(x, \lambda) = f(x) + r (-g(x))^2 + \lambda g(x) \rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda) \]
Optimization methods
Demonstration :: Augmented Lagrangian method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

\[ \textbf{Augmented Lagrangian method} \]

\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \rightarrow \min_x \max_{\lambda} \mathcal{L}(x, \lambda) \]

Additional unknown \( \lambda \)
Introduction to Optimization methods

Demonstration :: Augmented Lagrangian method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- **Augmented Lagrangian method**

\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \to \min_x \max_\lambda \mathcal{L}(x, \lambda) \]

- Standard Lagrangian \( r = 0 \)

- Isolines of the standard Lagrangian \( r = 0 \)
Optimization methods
Demonstration :: Augmented Lagrangian method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

**Augmented Lagrangian method**

\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \rightarrow \min_x \max_{\lambda} \mathcal{L}(x, \lambda) \]

Augmented Lagrangian \( r = 1 \)

Isolines of the augmented Lagrangian \( r = 1 \)
Optimization methods
Demonstration :: Augmented Lagrangian method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

**Augmented Lagrangian method**

\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda) \]

**Augmented Lagrangian**

\[ r = 10 \]

**Isolines of the augmented Lagrangian**

\[ r = 10 \]
Optimization methods
Demonstration :: Augmented Lagrangian method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- Augmented Lagrangian method

\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda) \]

Augmented Lagrangian \( r = 50 \)

Isolines of the augmented Lagrangian \( r = 50 \)
Optimization methods
Demonstration :: Augmented Lagrangian method

\[ f(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0 \]

- **Augmented Lagrangian method**

\[
\mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \rightarrow \min_x \max_\lambda \mathcal{L}(x, \lambda)
\]

**Advantages 😊**
- exact solution;
- smoothed functional.

**Drawbacks 😞**
- additional degrees of freedom increase the problem.
Optimization methods
Augmented Lagrangian method + Uzawa algorithm

- Augmented Lagrangian

\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \]

- Necessary conditions of the solution

\[
\begin{bmatrix}
\nabla_x \mathcal{L}(x, \lambda) \\
\nabla_\lambda \mathcal{L}(x, \lambda)
\end{bmatrix} = 0
\begin{bmatrix}
\nabla_x f(x) \\
0
\end{bmatrix} + \begin{bmatrix}
-2r \langle -g(x) \rangle \nabla_x g(x) \\
0
\end{bmatrix} + \begin{bmatrix}
\lambda \nabla g(x) \\
g(x)
\end{bmatrix}
\]

- Uzawa algorithm

\[ \lambda^{i+1} = \lambda^i - 2r \langle -g(x) \rangle \]
Optimization methods
Augmented Lagrangian method + Uzawa algorithm

- Augmented Lagrangian

\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \]

- Necessary conditions of the solution

\[
\begin{bmatrix}
\nabla_x \mathcal{L}(x, \lambda) \\
\nabla_\lambda \mathcal{L}(x, \lambda)
\end{bmatrix}
= 0
\begin{bmatrix}
\nabla_x f(x) \\
0
\end{bmatrix}
+ \begin{bmatrix}
-2r \langle -g(x) \rangle \nabla_x g(x) + \lambda \nabla_x g(x) \\
g(x)
\end{bmatrix}
\]

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Optimization methods
Augmented Lagrangian method + Uzawa algorithm

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\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \]

- Necessary conditions of the solution

\[
\begin{bmatrix}
\nabla_x \mathcal{L}(x, \lambda) \\
\nabla_\lambda \mathcal{L}(x, \lambda)
\end{bmatrix} = 0 \\
\begin{bmatrix}
\nabla_x f(x) \\
0
\end{bmatrix} + \begin{bmatrix}
-2r \langle -g(x) \rangle \nabla_x g(x) + \lambda \nabla_x g(x) \\
g(x)
\end{bmatrix}
\]

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Optimization methods
Augmented Lagrangian method + Uzawa algorithm

- Augmented Lagrangian

\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \]

- Necessary conditions of the solution

\[
\begin{bmatrix}
\nabla_x \mathcal{L}(x, \lambda) \\
\n\nabla_\lambda \mathcal{L}(x, \lambda)
\end{bmatrix} = 0 \begin{bmatrix}
\nabla_x f(x) \\
\n0
\end{bmatrix} + \begin{bmatrix}
(-2r \langle -g(x) \rangle + \lambda^i) \nabla_x g(x) \\
\n0
\end{bmatrix}
\]

- Uzawa algorithm

\[ \lambda^{i+1} = \lambda^i - 2r \langle -g(x) \rangle \]
Optimization methods
Augmented Lagrangian method + Uzawa algorithm

- Augmented Lagrangian

$$\mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x)$$

- Necessary conditions of the solution

$$\begin{bmatrix} \nabla_x \mathcal{L}(x, \lambda) \\ \nabla_\lambda \mathcal{L}(x, \lambda) \end{bmatrix} = 0 \begin{bmatrix} \nabla_x f(x) \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda^{i+1} \nabla_x g(x) \\ g(x) \end{bmatrix}$$

- Uzawa algorithm

$$\lambda^{i+1} = \lambda^i - 2r \langle -g(x) \rangle$$
Optimization methods
Augmented Lagrangian method + Uzawa algorithm

- Augmented Lagrangian

\[ \mathcal{L}(x, \lambda) = f(x) + r \langle -g(x) \rangle^2 + \lambda g(x) \]

- Uzawa algorithm

\[ \lambda^{i+1} = \lambda^i - 2r \langle -g(x) \rangle \]

Advantages ☻

- exact solution;
- smoothed functional;
- no additional degrees of freedom.

Drawbacks ☹
Plan

1. Introduction
2. Contact detection
3. Contact geometry
4. Contact discretization methods
5. Solution of contact problem
6. **Finite Element Analysis of contact problems**
7. Numerical examples
Introduction

- FEA requires
  - good finite element mesh
    - represents the real geometry;
    - fine enough to represent correctly stress-strain field;
    - rough enough to solve the problem in reasonable terms.
  - comprehension how close we are to the real solution;
  - careful apposition of boundary conditions.
Introduction

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FEA requires
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**Introduction**

- **FEA requires**
  - good finite element mesh
  - represents the real geometry;
  - fine enough to represent correctly stress-strain field;
  - rough enough to solve the problem in reasonable terms.

- comprehension how close we are to the real solution;
- careful apposition of boundary conditions.

Example of contact problem solved in ANSYS *(no FE mesh presented)*

Example of contact problem solved in ABAQUS *(no FE mesh presented)*
Convergence by mesh

Basics

One dimensional example on mesh refinement
Convergence by mesh

Basics

One dimensional example on mesh refinement

![Graph showing convergence by mesh refinement](image-url)
Convergence by mesh

Basics

One dimensional example on mesh refinement

![Diagram showing mesh refinement with variable, coordinate, Dirichlet BC, Real solution, and Finite element mesh.](image)
Convergence by mesh

Basics

One dimensional example on mesh refinement

Variable

Real solution

FE solution

Finite element mesh

Coordinate
One dimensional example on mesh refinement

Convergence by mesh
Basics
One dimensional example on mesh refinement
One dimensional example on mesh refinement

$V_{\text{max}}$

Real solution

Finite element mesh

Coordinate

Variable
Convergence by mesh
Basics

One dimensional example on mesh refinement

![Diagram showing one dimensional example on mesh refinement. The diagram includes a graph with a variable axis and a coordinate axis. The graph shows the relationship between the variable and the coordinate, with a peak indicated by $V_{max}$ and a real solution indicated.]
One dimensional example on mesh refinement

Smart meshing with linear elements
One dimensional example on mesh refinement

Variable

Coordinate

Quadratic finite element mesh

Real solution

$V_{max}$

Smart meshing with quadratic elements
One dimensional example on mesh refinement

Case of singularity
Convergence by mesh
Basics

One dimensional example on mesh refinement

Variable

Coordinate

Finite element mesh

Real solution

Case of hidden maximum

V_{max}
One dimensional example on mesh refinement

Convergence by mesh
Convergence by mesh

One dimensional example on mesh refinement

Real solution

FE solution

Vmax

Number of nodes

Convergence by mesh
Convergence by mesh

Basics

One dimensional example on mesh refinement

No convergence by mesh (singularity)
Convergence by mesh
Basics

One dimensional example on mesh refinement

Second dangerous point

First dangerous point

Case of two maximums
Boundary conditions
How fast can we go?

General thinks

- Contact problems are always nonlinear
- Nonlinear problems require slow change of boundary conditions
  - infinite looping;
  - convergence to the wrong solution.
Boundary conditions
How fast can we go?

General thinks
- Contact problems are always nonlinear
- Nonlinear problems requires slow change of boundary conditions

Resolution of nonlinear problem

Departure point $R(x_0, f_0) = 0$
General thinks

- Contact problems are always nonlinear
- Nonlinear problems requires slow change of boundary conditions

Resolution of nonlinear problem

Change of boundary conditions $R(x_0, f_1) \neq 0$
General thinks

- Contact problems are always nonlinear
- Nonlinear problems requires slow change of boundary conditions

**Resolution of nonlinear problem**

\[ R(x_0, f_1) + \left. \frac{\partial R}{\partial x} \right|_{x_0} \delta x = 0 \rightarrow \delta x = - \left. \frac{\partial R}{\partial x} \right|_{x_0} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x \]
General thinks

- Contact problems are always nonlinear
- Nonlinear problems require slow change of boundary conditions

Resolution of nonlinear problem

\[ R(x^1, f_1) + \left. \frac{\partial R}{\partial x} \right|_{x^1} \delta x = 0 \rightarrow \delta x = - \left. \frac{\partial R}{\partial x} \right|_{x^1} R(x^1, f_1) \rightarrow x^2 = x^1 + \delta x \]
Boundary conditions
How fast can we go?

General thinks
- Contact problems are always nonlinear
- Nonlinear problems requires slow change of boundary conditions

Resolution of nonlinear problem

Convergence

$$\|x^{i+1} - x^i\| \leq \varepsilon \rightarrow x_1 = x^{i+1}$$
Boundary conditions
How fast can we go?

General thinks

- Contact problems are always nonlinear
- Nonlinear problems requires slow change of boundary conditions

Infinite looping

Departure point $R(x_0, f_0) = 0$
Boundary conditions
How fast can we go?

General thinks
- Contact problems are always nonlinear
- Nonlinear problems require slow change of boundary conditions

Infinite looping

Too fast change of boundary conditions $R(x_0, f_1) \neq 0$
General thinks

- Contact problems are always nonlinear
- Nonlinear problems requires slow change of boundary conditions

\[
R(x_0, f_1) + \frac{\partial R}{\partial x} \bigg|_{x_0} \delta x = 0 \rightarrow \delta x = -\frac{\partial R}{\partial x} \bigg|_{x_0} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x
\]
Boundary conditions

How fast can we go?

General thinks

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Infinite looping

Newton-Raphson iterations

\[ R(x^1, f_1) + \left. \frac{\partial R}{\partial x} \right|_{x^1} \delta x = 0 \rightarrow \delta x = - \left. \frac{\partial R}{\partial x} \right|_{x^1} R(x^1, f_1) \rightarrow x^2 = x^1 + \delta x \]
General thinks

- Contact problems are always nonlinear
- Nonlinear problems require slow change of boundary conditions

![Infinite looping diagram]

Infinite looping
**Boundary conditions**

*How fast can we go?*

---

**General thinks**

- Contact problems are always nonlinear
- Nonlinear problems requires slow change of boundary conditions

---

**Convergence to the wrong solution**

Departure point $R(x_0, f_0) = 0$
Boundary conditions

How fast can we go?

General thinks

- Contact problems are always nonlinear
- Nonlinear problems require slow change of boundary conditions

Convergence to the wrong solution

Too fast change of boundary conditions $R(x_0, f_1) \neq 0$
Boundary conditions
How fast can we go?

General thinks
- Contact problems are always nonlinear
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Convergence to the wrong solution

Newton-Raphson iterations

\[ R(x_0, f_1) + \frac{\partial R}{\partial x} \bigg|_{x_0} \delta x = 0 \rightarrow \delta x = -\frac{\partial R}{\partial x} \bigg|_{x_0} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x \]
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Convergence to the wrong solution

Newton-Raphson iterations

\[ R(x^1, f_1) + \left. \frac{\partial R}{\partial x} \right|_{x^1} \delta x = 0 \rightarrow \delta x = - \left. \frac{\partial R}{\partial x} \right|_{x^1} R(x^1, f_1) \rightarrow x^2 = x^1 + \delta x \]
Boundary conditions
How fast can we go?

General thinks
- Contact problems are always nonlinear
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Convergence to the wrong solution

Solution. Correct?
Methods passing Taylor’s patch test

- mortar;
- Nitsche;
- node-to-node;
- Contact domain method.

Possible schemes for contact patch test
Patch test

Methods passing Taylor’s patch test
- mortar;
- Nitsche;
- node-to-node;
- Contact domain method.

Method not passing Taylor’s patch test
- node-to-segment.

NTS not passing patch test - oscillation of contact pressure (top)
Nitsche method passing patch test
Patch test

Methods passing Taylor’s patch test
- mortar;
- Nitsche;
- node-to-node;
- Contact domain method.

Method not passing Taylor’s patch test
- node-to-segment.

But!
- NTS passes the patch test in two pass;
- NTS passing patch test [G. Zavarise, L. De Lorenzis, 2009];
- Revisiting Taylor’s patch test [Crisfield].

NTS not passing patch test - oscillation of contact pressure (top)
Nitsche method passing patch test
Master-slave discretization

General rules

**Rule 1**

- Contacting surface with higher mesh density is always slave surface.

**Rule 1**

- If the mesh densities are equal on two surfaces, master surface is the surface which deforms less.

---

**Initial FE mesh**

**Incorrect master-slave choice 😞**

**Correct master-slave choice 😊**
Plan

1. Introduction
2. Contact detection
3. Contact geometry
4. Contact discretization methods
5. Solution of contact problem
6. Finite Element Analysis of contact problems
7. Numerical examples
Validation

3D contact

Increments
Validation
3D contact

Increments
Validation

3D contact

Increments
Validation
Shallow ironing

Finite element mesh

Description
- Plane strain
- $E_i = 68.96 \cdot 10^8$ Pa
- $E_s = 68.96 \cdot 10^7$ Pa
- $\nu_i = \nu_s = 0.32$
- $\mu = 0.3$
- $\Delta u_v = 1\text{mm}/10\text{ incr}$
- $\Delta u_h = 10\text{mm}/500\text{incr}$
- $NN = 3840$
Validation

Shallow ironing

Results $\langle \text{Stress}_{12} \rangle$
Validation

Shallow ironing

Results $\langle \text{Stress}_{12} \rangle$
Validation
Shallow ironing

Results $\sigma_{12}$
Validation
Shallow ironing

Results $\text{Stress}_{12}$
Validation
Shallow ironing

Results $<\text{Stress}_{12}>$
Validation
Shallow ironing

Results $\langle \text{Stress}_{12} \rangle$
Validation
Shallow ironing

Comparison

- K.A. Fischer, P. Wriggers [2006]
Validation
Shallow ironing

Comparison

- J. Oliver, S. Hartmann [2009]
Validation
Shallow ironing

Comparison

- Our results [2009]
Validation
Shallow ironing

Comparison

- Our results [2009]
Validation
Klang’s problem

Finite element mesh

Description
- Plane stress
- $E = 2.1 \cdot 10^{11}$ Pa
- $\nu = 0.3$
- $\mu = 0.4$
- $r = 5.999$ cm
- $R = 6$ cm
- $F = 18750$ N
- $\alpha = 120^\circ$
- $NN = 2500$

Plane stress

$E = 2.1 \cdot 10^{11}$ Pa

$\nu = 0.3$

$\mu = 0.4$

$r = 5.999$ cm

$R = 6$ cm

$F = 18750$ N

$\alpha = 120^\circ$

$NN = 2500$
Validation

Klang’s problem

Finite element mesh
Validation
Klang’s problem

Finite element mesh
Validation
Klang’s problem

Results $\langle \text{Stress}_{22} \rangle$
Validation
Klang’s problem

Results $\langle \text{Stress}_{22} \rangle$
Validation

Klang’s problem

Results $\text{Stress}_{22}$
Validation

Klang’s problem

Results $\langle \text{Stress}_{22} \rangle$
Validation
Klang's problem

Results \langle \text{Stress}_{22} \rangle
Validation
Klang’s problem

Results

Validation
Klang’s problem

Results

ZEBULON, 1 increment

Klang’s problem, 1 increment

Shear contact stress distribution along the contact surface

angle, degrees

Shear contact stress distribution along the contact surface

ZEBULON, sticking zone

ZEBULON, sliding zone

Klang’s analytical solution

Simulation P. Alart et A. Curnier
Validation
Klang’s problem

Results

ZEBULON, 5 increment

Klang’s problem, 5 increments

Shear contact stress distribution along the contact surface

angle, degrees

Klang’s analytical solution
Simulation P. Alart et A. Curnier
ZEBULON, sticking zone
ZEBULON, sliding zone
Validation
Klang’s problem

Results

ZEBULON, 10 increment

Klang’s problem. 10 increments

Shear contact stress distribution along the contact surface

angle, degrees

0 10 20 30 40 50 60

0 100 200 300 400 500 600 700 800

Klang’s analytical solution
Simulation P. Alart et A. Curnier
ZEBULON, sticking zone
ZEBULON, sliding zone
Validation
Klang’s problem

Results

ZEBULON, 25 increment

Klang’s problem. 25 increments

- Klang’s analytical solution
- Simulation P.A. Alart et A. Curnier
- ZEBULON, sticking zone
- ZEBULON, sliding zone

Shear contact stress distribution along the contact surface

angle, degrees

0 10 20 30 40 50 60

0 100 200 300 400 500 600 700 800
Validation
Klang’s problem

Results

ZEBULON, 50 increment

Klang’s problem. 50 increments

Shear contact stress distribution along the contact surface

angle, degrees

0 10 20 30 40 50 60

0 100 200 300 400 500 600 700 800

Klang’s analytical solution
Simulation P.Alart et A.Curnier
ZEBULON, sticking zone
ZEBULON, sliding zone
Validation

Klang’s problem

Results

ZEBULON, 100 increment

Klang’s problem, 100 increments

Shear contact stress distribution along the contact surface

angle, degrees

0 10 20 30 40 50 60

0 100 200 300 400 500 600 700 800

Klang’s analytical solution
Simulation P. Alart et A. Curnier
ZEBULON, sticking zone
ZEBULON, sliding zone
Performance
Disk-blade contact

Disk-blade frictional contact, elasto-plastic material
Performance
Multi contact

Multi plate frictionless contact
Performance

Multi contact

Multi plate frictionless contact
Multi plate frictionless contact
Multi plate frictionless contact
Performance

Multi contact

Multi plate frictionless contact
Multi plate frictionless contact
Performance

Multi contact

Multi plate frictionless contact
Performance

Multi contact

Multi plate frictionless contact