Non linear finite element method

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Introduction — Outline

- The finite element method
- Application to mechanics
- Solving systems of non linear equations
- Incompressibility

Recalls about the finite element method

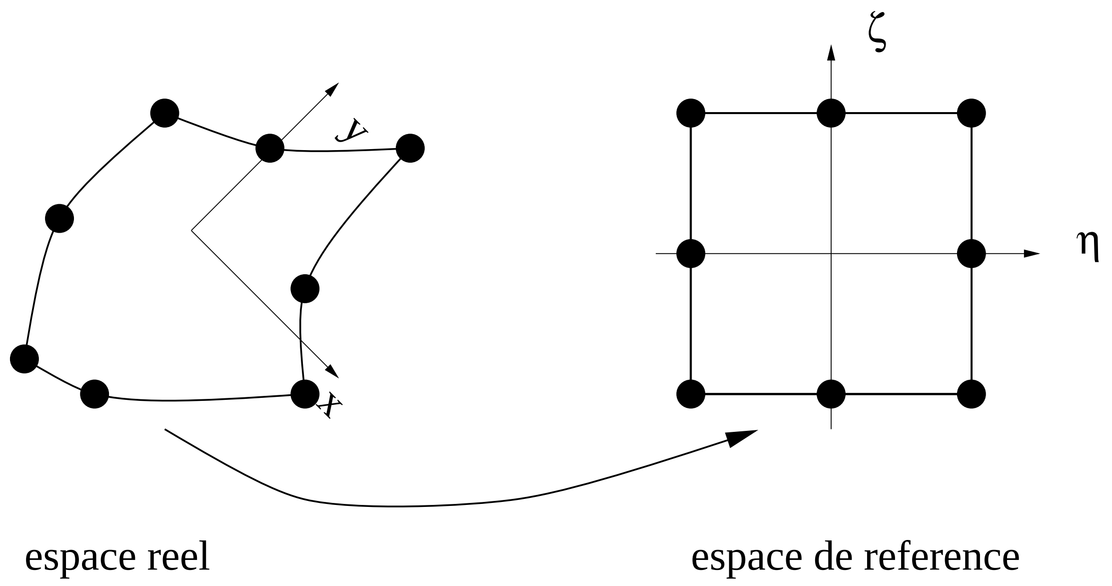
Spatial discretisation

nodes, edges, faces (3D), elements

Node position

actual coordinates $\underline{\mathbf{x}} = (x, y, z)$

reference coordinates $\underline{\eta} = (\eta, \zeta, \xi)$



Coordinates of the nodes belonging to one element:

$$\underline{\mathbf{x}}^i,\ i = 1 \dots N$$

$$\underline{\mathbf{x}} = \sum_i N^i(\underline{\eta})\, \underline{\mathbf{x}}^i$$

N^i interpolation function (or shape functions) such that:

$$N^i(\underline{\eta}_j) = \delta_{ij} \quad \text{et} \quad \sum_i N^i(\underline{\eta}) = 1,\ \forall \underline{\eta}$$

Jacobian matrix of the transformation $\underline{\eta} \rightarrow \underline{\mathbf{x}}(\underline{\eta})$

$$\underset{\sim}{\mathbf{J}} = \frac{\partial \underline{\mathbf{x}}}{\partial \underline{\eta}}$$

$$J_{ij} = \frac{\partial x_i}{\partial \eta_j} = \frac{\partial (N^k x_i^k)}{\partial \eta_j} = x_i^k \frac{\partial N^k}{\partial \eta_j}$$

Jacobian:

$$J = \det(\underset{\sim}{\mathbf{J}})$$

Discrete integration methods

- Gauss method 1D

$$\int_{-1}^{1} f(x)dx = w^i f(x^i)$$

x_i	positions where the function is evaluated
w^i	weight associated to the Gauss points

A Gauss integration with n Gauss points can exactly evaluate the integral of a $2n-1$ order polynom.

- Extension to 2D and 3D cases.

Gauss points are very important in the case of the non linear FEM as the material behavior is evaluated at each Gauss point. State variables must be stored for each Gauss point

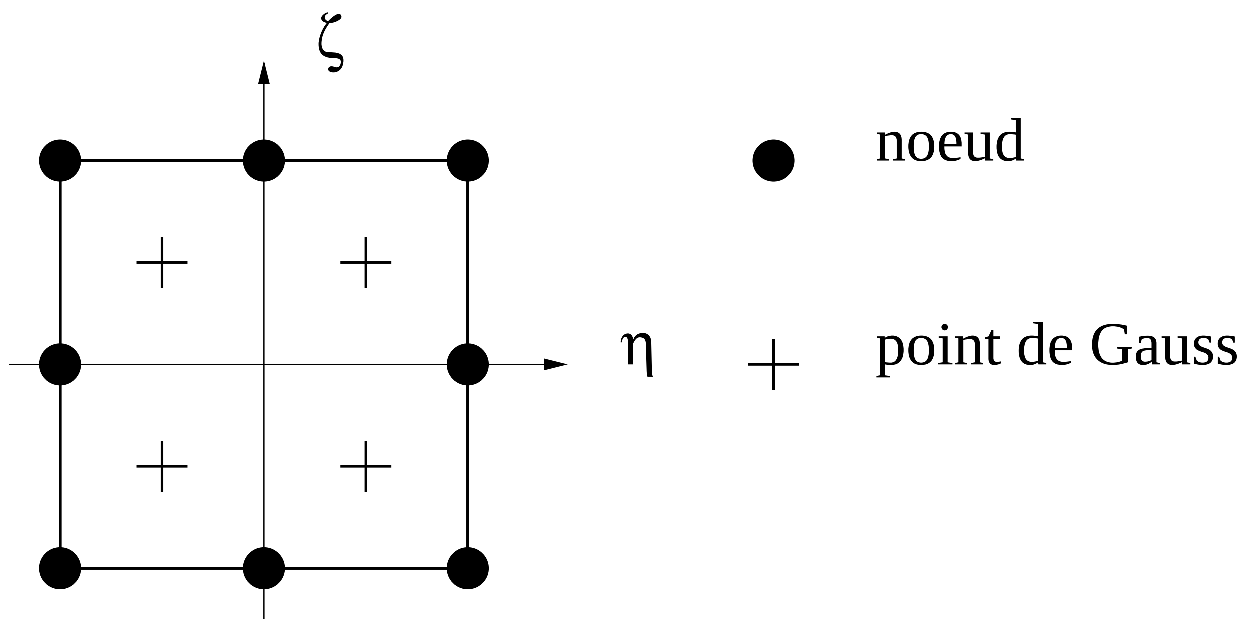
Integral over one finite element V_e (reference element V_r)

$$\int_{V_e} f(\underline{\mathbf{x}})dx = \int_{V_r} f(\underline{\eta})Jd\eta = \sum_i f(\underline{\eta}^i)(Jw^i)$$

It is possible to define the volume associated to a given Gauss point i:

$$v^i = Jw^i$$

Discretisation of unknown fields



The unknown fields (displacement, temperature, pressure,. . .) are discretized in order to solve the problem:

at nodes : displacement, temperature, pressure

at elements: pressure in the case of incompressible materials

at element sets : periodic elements (homogeneosation), 2D elements inclusing torsion/flexion on the third direction

• Mechanics ($\underline{\mathbf{u}}^i$ displacement at node i)

$$\underline{\mathbf{u}}(\underline{\eta}) = N^k(\underline{\eta})\underline{\mathbf{u}}^k$$

• Thermal problem (T : temperature) :

$$T(\underline{\eta}) = N^i(\underline{\eta})T^i$$

• Computation of the gradients:

$$(\underset{\sim}{\mathrm{grad}}\,\underline{\mathbf{u}})_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial \eta_k}\frac{\partial \eta_k}{\partial x_i} = u_i^n \frac{\partial N^n}{\partial \eta_k}\frac{\partial \eta_k}{\partial x_i}$$

$$(\underline{\mathrm{grad}}T)_i = \frac{\partial T}{\partial x_i} = \frac{\partial T}{\partial \eta_k}\frac{\partial \eta_k}{\partial x_i} = T^n \frac{\partial N^n}{\partial \eta_k}\frac{\partial \eta_k}{\partial x_i}$$

These formula can be rewritten in a compact form as:

$$\left\{\underset{\sim}{\mathrm{grad}}\,\underline{\mathbf{u}}\right\} = [B]\,.\,\{u\}$$

$$\underline{\mathrm{grad}}T = [A]\,.\,\{T\}$$

$\{u\}$ and $\{T\}$ are vectors of nodal variables:

$$\{u\} = \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ \vdots \\ u_1^N \\ u_2^N \\ u_3^N \end{Bmatrix} \qquad \{T\} = \begin{Bmatrix} T^1 \\ \vdots \\ T^N \end{Bmatrix}$$

Isoparametric elements Isoparametric elements are elements for which the unknowns and the coordinates are interpolated using the same shape functions.

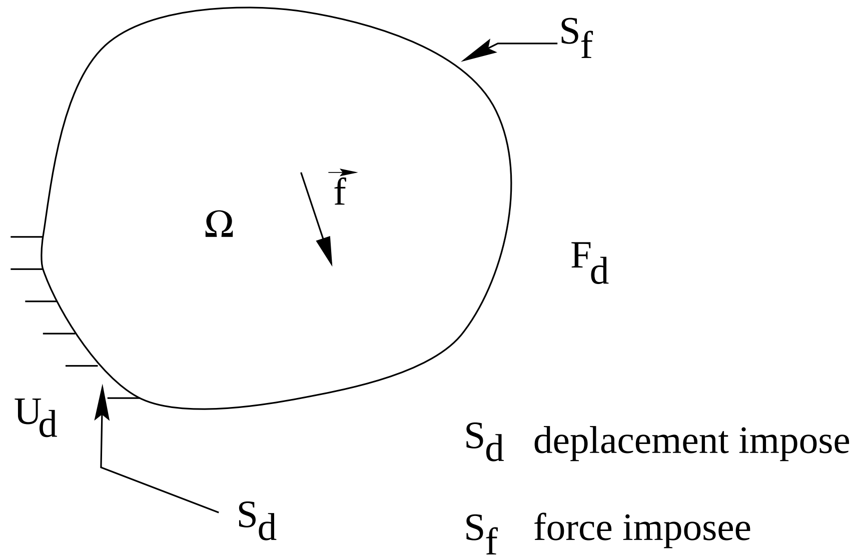
Voigt Notations

$$\left\{\underset{\sim}{\text{grad}}\,\underline{\mathbf{u}}\right\} = [B]\,.\,\{u\} \qquad ???$$

- The Voigt Notation is in fact used
- standard notation / recommanded notation :

$$\underset{\sim}{\varepsilon} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{pmatrix}, \quad \underset{\sim}{\sigma} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix} \quad \text{—} \quad \underset{\sim}{\mathbf{x}} = \begin{pmatrix} x_{11} \\ x_{22} \\ x_{33} \\ \sqrt{2}x_{12} \\ \sqrt{2}x_{23} \\ \sqrt{2}x_{31} \end{pmatrix}$$

Application to mechanics



In the static case, the problem to be solved is as follows:

$$\underline{\mathrm{div}}\,\underset{\sim}{\sigma} + \underline{\mathbf{f}} = \underline{\mathbf{0}} \qquad \text{sur } \Omega$$
$$\underset{\sim}{\sigma}.\underline{\mathbf{n}} = \underline{\mathbf{T}} \qquad \text{sur } S_f$$
$$\underline{\mathbf{u}} = \underline{\mathbf{u}}^d \qquad \text{sur } S_d$$

where Ω is the volume of the solid, S_d surfaces where displacements are imposed, S_f surfaces where efforts are imposed. $\underline{\mathbf{f}}$ represents bodu forces (i.e. gravity)

Principle of virtual work

- Statically admissible stress field: A stress field $\underset{\sim}{\sigma}^*$ is statically admissible if:

$$\underline{\mathrm{div}}\underset{\sim}{\sigma}^* + \underline{\mathbf{f}} = \underline{\mathbf{0}} \qquad \text{on } \Omega$$
$$\underset{\sim}{\sigma}^*.\underline{\mathbf{n}} = \underline{\mathbf{F}}_d \qquad \text{on } S_f$$

- Kinematically admissible displacement field: A displacement field $\underline{\mathbf{u}}'$ is kinematically admissible if:

$$\underline{\mathbf{u}}' = \underline{\mathbf{u}}^d \qquad \text{sur } S_d$$

- Principle of virtual work: Let $\underset{\sim}{\sigma}^*$ be a statically admissible stress field and let $\underline{\mathbf{u}}'$ be a kinematically admissible displacement field

$$\int_\Omega \underset{\sim}{\sigma}^* : \underset{\sim}{\epsilon}' d\Omega = \int_\Omega \underline{\mathbf{f}}.\underline{\mathbf{u}}' d\Omega + \int_S \underline{\mathbf{T}}.\underline{\mathbf{u}}' dS$$

$$\underset{\sim}{\epsilon}' = \frac{1}{2}\left((\underset{\sim}{\mathrm{grad}}\,\underline{\mathbf{u}}') + (\underset{\sim}{\mathrm{grad}}\,\underline{\mathbf{u}}')^T\right)$$

Le left hand side corresponds to the internal virtual work and the right handside to the external virtual work.

Equilibrium (MEF)

• The discretized displacement field is KA

• The associated stress field is not necessarily SA ($\underset{\sim}{\sigma}(\underset{\sim}{\varepsilon}(\vec{u}))$)

• Solving the problem: Find the displacement fiel such that the associated stress field verifies the PVW.

Discretisation of the PVW : internal virtual work

$$
\begin{aligned}
w_i &\equiv \{F_i(\{u\})\} \,.\, \{\dot{u}'\} \\
&= \int_\Omega \underset{\sim}{\sigma}(\underline{\mathbf{u}}) : \underset{\sim}{\dot{\varepsilon}}(\underline{\dot{\mathbf{u}}}') dV \\
&= \sum_e \int_{V_e} \{\underset{\sim}{\sigma}(\{u^e\})\} \,.\, [B] \,.\, \{\dot{u}^{e\prime}\} \, dV \\
&= \sum_e \left(\int_{V_e} [B]^T \,.\, \{\underset{\sim}{\sigma}(\{u^{e\prime}\})\} \, dV \right) . \, \{\dot{u}^{e\prime}\} \\
&= \sum_e \{F_i^e\} \,.\, \{\dot{u}^{e\prime}\}
\end{aligned}
$$

$$
\{\sigma\} \,.\, [B] \,.\, \{\dot{u}^{e\prime}\} = \sigma_i B_{ij} \dot{u}_j^{e\prime} = \sigma_i B_{ji}^T \dot{u}_j^{e\prime} = \left(B_{ji}^T \sigma_i\right) \dot{u}_j^{e\prime}
$$

Discretisation of the PVW : external virtual work (imposed volume forces)

$$
\begin{aligned}
w_e &\equiv \{F_e(\{u\})\} . \{\dot{u}'\} \\
&= \int_\Omega \underline{\mathbf{f}}.\underline{\dot{\mathbf{u}}}' dV \\
&= \sum_e \int_{V_e} \underline{\mathbf{f}}. [N] . \{\underline{\dot{\mathbf{u}}}^{e\prime}\} \, dV \\
&= \sum_e \left(\int_{V_e} [N]^T . \underline{\mathbf{f}} dV \right) \{\underline{\dot{\mathbf{u}}}^{e\prime}\} \\
&= \sum_e \{F_e^e\} . \{\dot{u}^{e\prime}\}
\end{aligned}
$$

Resolution

$$
\begin{aligned}
w_i &= w_e \\
\Rightarrow \quad \{F_i(\{u\})\} . \{\dot{u}'\} &= \{F_e(\{u\})\} . \{\dot{u}'\} \quad \forall \{\dot{u}'\} \\
\Rightarrow \quad \{F_i(\{u\})\} &= \{F_e(\{u\})\}
\end{aligned}
$$

This system can be solved using an iterative Newton method (in the following) which requires the calculation of:

$$
[K] = \frac{\partial \{F_i(\{u\})\}}{\partial \{u\}}
$$

Note that

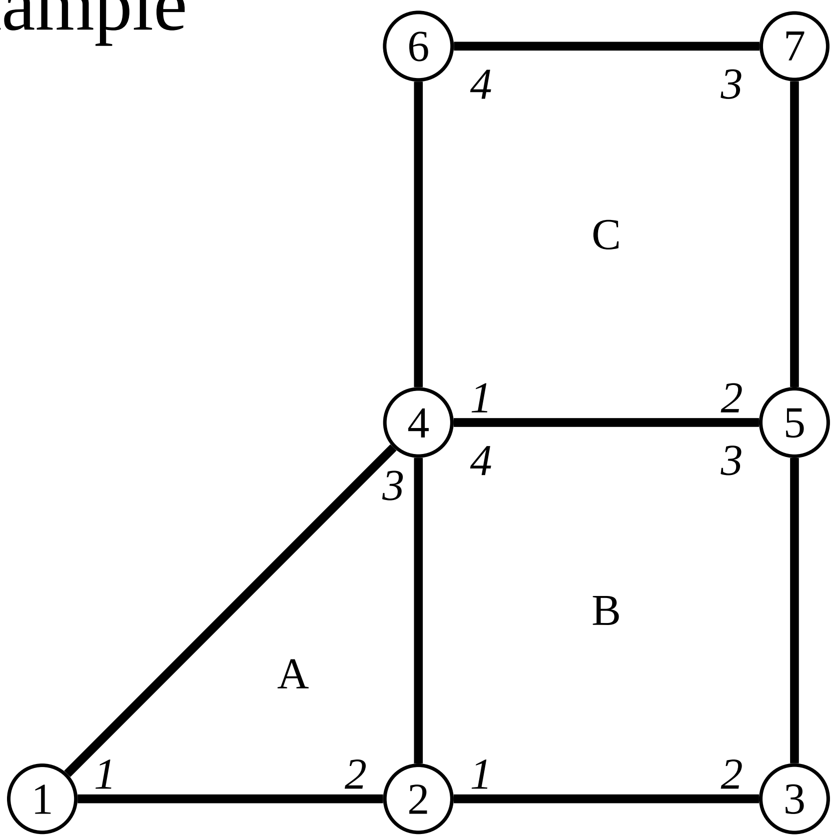
$$
\begin{aligned}
[K^e] &= \frac{\partial \{F_i^e(\{u^e\})\}}{\partial \{u^e\}} \\
&= \int_{V_e} [B]^T . \frac{\partial \{\underset{\sim}{\sigma}\}}{\partial \{\underset{\sim}{\varepsilon}\}} . \frac{\partial \{\underset{\sim}{\varepsilon}\}}{\partial \{u^e\}} dV = \int_{V_e} [B]^T . \left[\underset{\approx}{\mathbf{L}}_c \right] . [B] \, dV
\end{aligned}
$$

Assembly of the global stiffness matrix

The global stiffness matrix is obtained by assembling the $[K^e]$ matrices.

The internal forces vector $\{F_i\}$ ($\{F_e\}$) is obtained by assembling the $\{F_i^e\}$ ($\{F_e^e\}$) vectors.

Example



$$\{u\} = \{u_x^1, u_y^1, u_x^2, u_y^2, u_x^3, u_y^3, u_x^4, u_y^4, u_x^5, u_y^5, u_x^6, u_y^6, u_x^7, u_y^7\}$$

For element A, the local unknown vector $\{u^A\}$is:

$$\{u^A\} = \{u_x^{A1}, u_y^{A1}, u_x^{A2}, u_y^{A2}, u_x^{A3}, u_y^{A3}\} = \{u_x^1, u_y^1, u_x^2, u_y^2, u_x^4, u_y^4\}$$

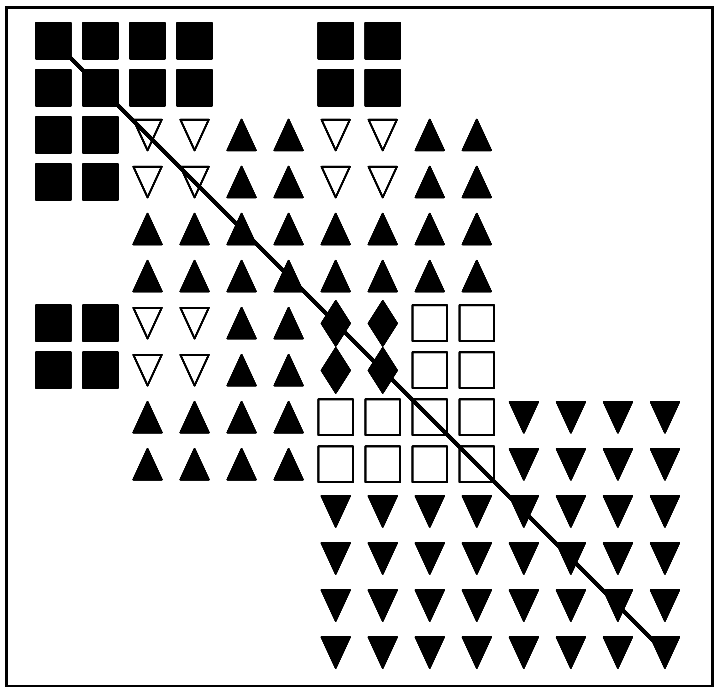
The associated internal forces vector associated to element A is:

$$\{F_i^A\} = \{F_x^{A1}, F_y^{A1}, F_x^{A2}, F_y^{A2}, F_x^{A3}, F_y^{A3}\}$$

Internal Forces

$$\{F_i\} = \left\{ \begin{array}{lcl} F_x^1 & = & F_x^{A1} \\ F_y^1 & = & F_y^{A1} \\ F_x^2 & = & F_x^{A2} + F_x^{B1} \\ F_y^2 & = & F_y^{A2} + F_y^{B1} \\ F_x^3 & = & F_x^{B2} \\ F_y^3 & = & F_y^{B2} \\ F_x^4 & = & F_x^{A3} + F_x^{B4} + F_x^{C1} \\ F_y^4 & = & F_y^{A3} + F_y^{B4} + F_y^{C1} \\ F_x^5 & = & F_x^{B3} + F_x^{C2} \\ F_y^5 & = & F_y^{B3} + F_y^{C2} \\ F_x^6 & = & F_x^{C4} \\ F_y^6 & = & F_y^{C4} \\ F_x^7 & = & F_x^{C3} \\ F_y^7 & = & F_y^{C3} \end{array} \right\}$$

Stiffness matrix

$[K] =$ 

■	A
▲	B
▼	C
▽	A and B
△	A and C
□	B and C
♦	A, B and C

Incremental resolution
Small strain case

- Strong non linearity $\Rightarrow$ incremental resolution
- current time increment: from t_0 to t_1, $\Delta t = t_1 - t_0$
- Many increments may be needed
- Quantities $\{F_i\}$, $\{F_e\}$ and $[K_T]$ are computed for each element and assembled. For instancen the internal forces foe element e are computed as:

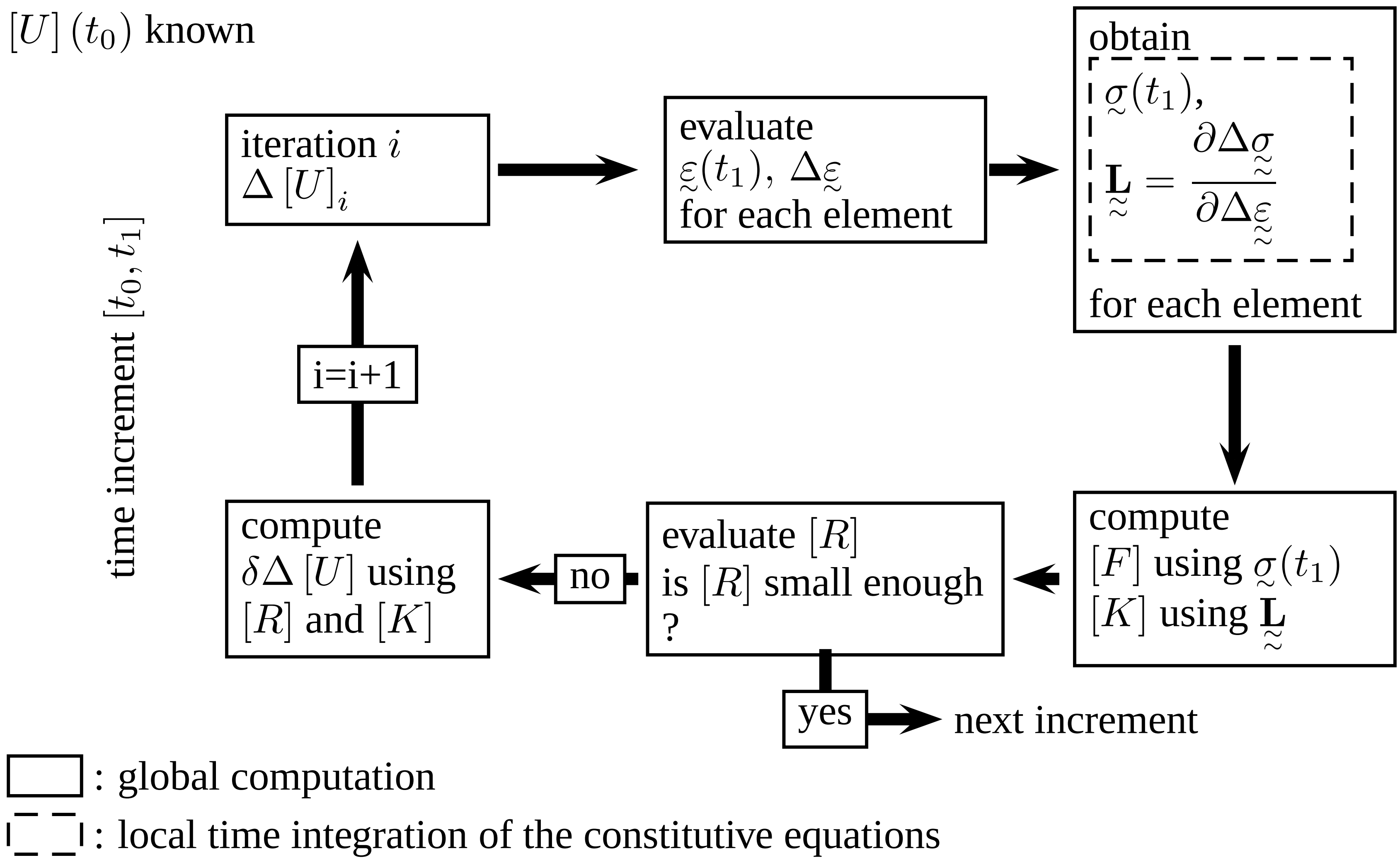
$$\{F_i^e\} = \int_{V_e} [B]^T . \{\sigma\} \, dV = \sum_g \left[B_g^T\right] . \{\sigma_g\} \left(J_g w_g\right)$$

The elementaty stiffness matrix is computed as:

$$[K^e] = \int_{V_e} [B]^T . [L] . [B] \, dV = \sum_g [B_g]^T . [L] . [B_g] \left(J_g w_g\right)$$

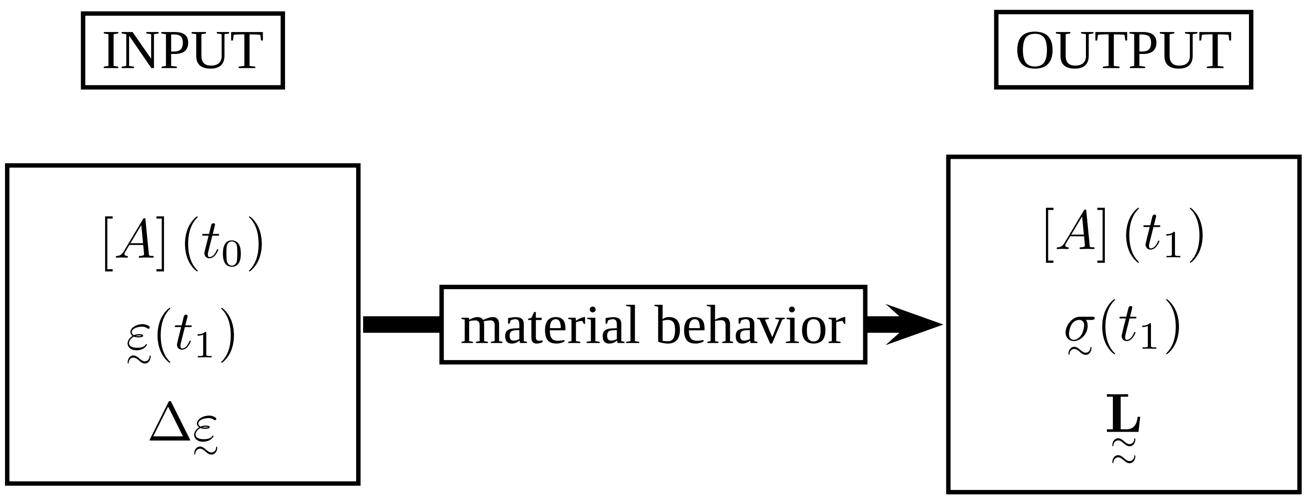
- Once assembled vectors $\{F_i\}$ and $\{F_e\}$ are vectors whose size is the number of unknown quantities (n_d). $[K]$ is a $n_d \times n_d$ matrix.

Algorithm for the resolution
Material behavior in the finite element method



A generic interface between the material behavior and the FEM

- $\Delta t = t_1 - t_0$



Numerical methods to solve not linear systems of equations

Non linear equations written as:

$$\{R\}\left(\{U\}\right) = \{0\}$$

FEM case:

$$\{F_i\}\left(\{u\}\right) - \{F_e\}\left(\{u\}\right) = \{0\}$$

Newton methods

Linearisation of the system $\{R\}(\{U\}) = \{0\}$:

$$\{R\}(\{U\}) = \{R\}(\{U\}_k) + \left.\frac{\partial \{R\}}{\partial \{U\}}\right|_{\{U\}=\{U\}_k} (\{U\} - \{U\}_k)$$

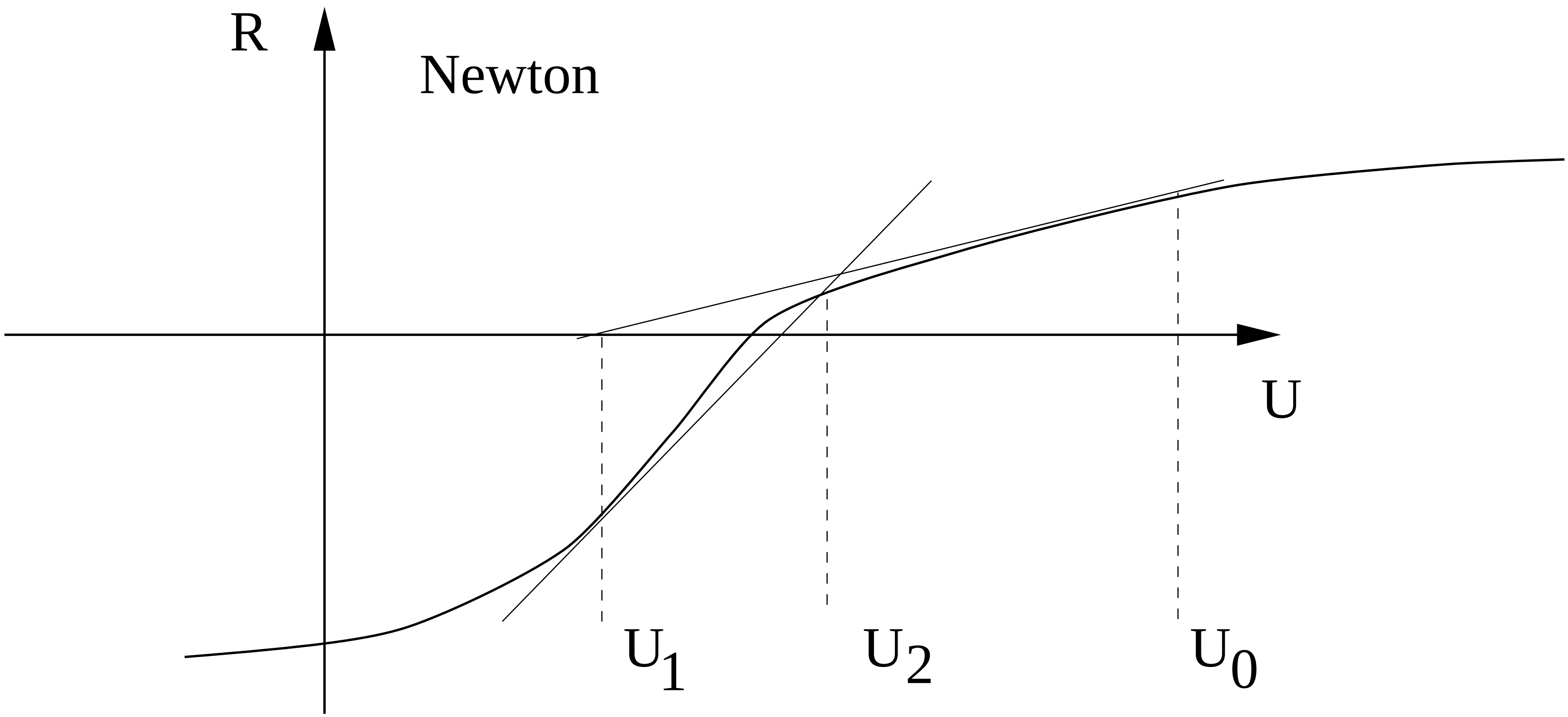
(k : iteration number) Notation

$$[K](\{U\}) = \frac{\partial \{R\}}{\partial \{U\}} \qquad K_{ij}(\{U\}) = \frac{\partial R_i}{\partial U_j}$$

After resolution

$$\{U\}_{k+1} = \{U\}_k - [K]_k^{-1} \{R\}_k$$

Illustration



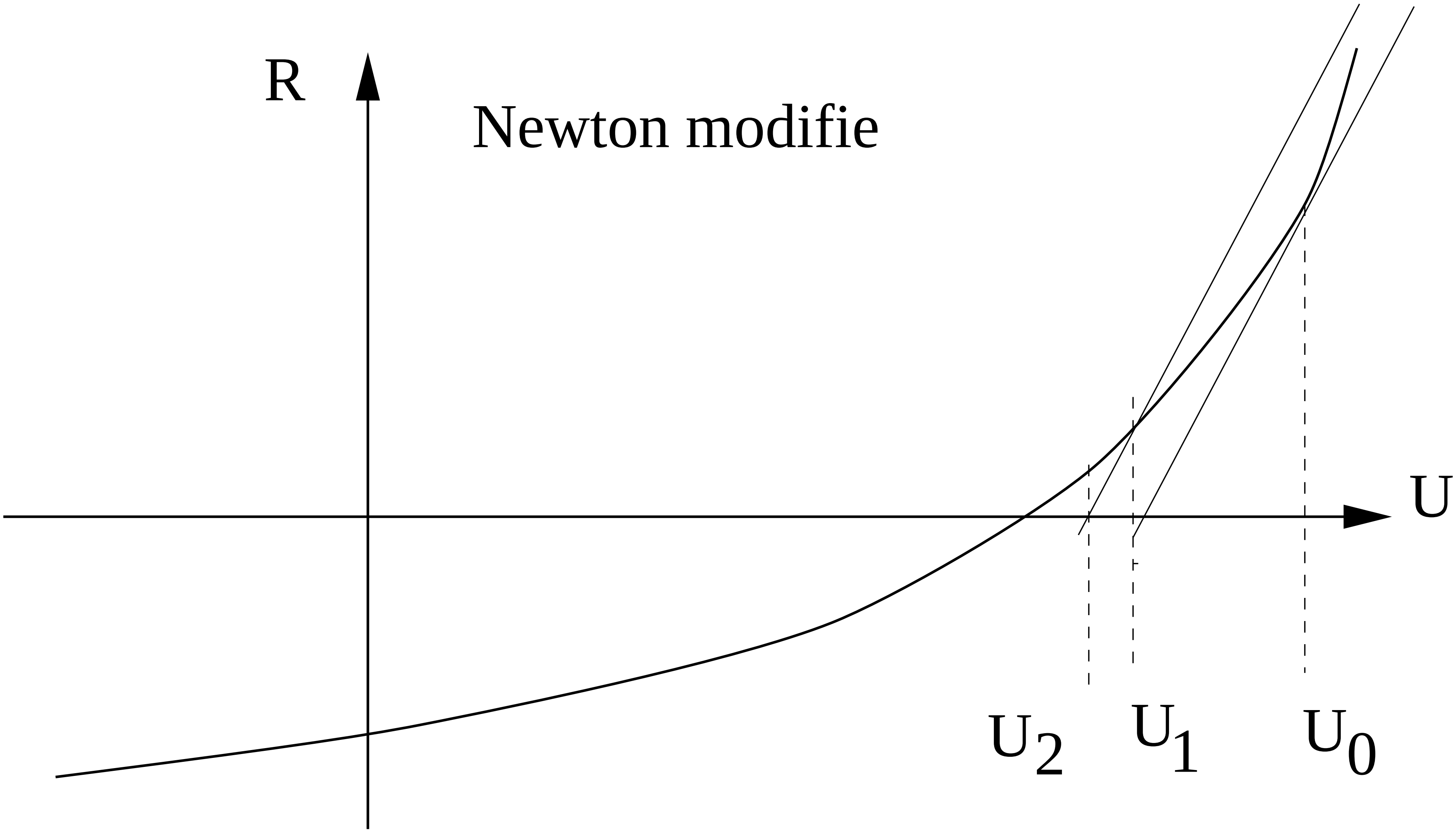
Quasi Newton methods

When the number of unknowns is large:

$$\text{computational cost of } \; [K]^{-1} \gg \text{computational cost of } \; \{R\}(\{U\}) \quad \text{and} \quad [K]^{-1}\{R\}_k$$

The inverse matrix computed at the first iteration is kept:

$$\{U\}_{k+1} = \{U\}_k - [K]_0^{-1}\{R\}_k$$


R
Newton modifie
U
U2
U1
U0

Other possibilities:

$$\begin{aligned}
\{U\}_1 &= \{U\}_0 - [K]_0^{-1}\{R\}_0 \\
\{U\}_2 &= \{U\}_1 - [K]_1^{-1}\{R\}_1 \\
\{U\}_{k+1} &= \{U\}_k - [K]_1^{-1}\{R\}_k
\end{aligned}$$

One unknown case : order of convergence

Fixed Point Method

to be solved (x is scalar):

$$f(x) = 0$$

Transformation:

$$x = g(x)$$

Solution : **fixed point** .

Iterative resolution. x_0 is given.

$$x_{n+1} = g(x_n)$$

Let s be the solution of $x = g(x)$.

> If there exits an interval around s such that $|g'| \leq K < 1$ then the x_n serie converges toward s.

To prove this, one first notices that there exists value t ($t \in [x, s]$) such that (Mean Value Theorem)

$$g\left(x\right) - g\left(s\right) = g'\left(t\right)\left(x - s\right)$$

as $g\left(s\right) = s$ et $x_n = g\left(x_{n-1}\right)$, one gets :

$$\begin{aligned} |x_n - s| = |g\left(x_{n-1}\right) - g\left(s\right)| \qquad & \leq |g'\left(t\right)||x_{n-1} - s| \\ & \leq K|x_{n-1} - s| \\ & \leq \cdots \leq K^n|x_0 - s| \end{aligned}$$

as $K < 1$, $\lim_{\rightarrow\infty} |x_n - s| = 0$.

Order of an iterative method

ϵ_n error on x_n

$$x_n = s + \epsilon_n$$

The Taylor expansion of x_{n+1} leads to

$$\begin{aligned} x_{n+1} = g\left(x_n\right) &= g(s) + g'(s)\left(x_n - s\right) + \frac{1}{2}g''(s)\left(x_n - s\right)^2 \\ &= g(s) + g'(s)\epsilon_n + \frac{1}{2}g''(s)\epsilon_n^2 \end{aligned}$$

One then gets

$$x_{n+1} - g(s) = x_{n+1} - s = \epsilon_{n+1} = g'(s)\epsilon_n + \frac{1}{2}g''(s)\epsilon_n^2$$

The Order of an iterative method gives a mesure of its convergence rate. At orde 1 one gets

$$\epsilon_{n+1} \approx g'(s)\epsilon_n$$

and at order 2

$$\epsilon_{n+1} \approx \frac{1}{2}g''(s)\epsilon_n^2$$

Application to the Newton method

In the case of the Newton method, a Taylor expansion around x_n is used to find x_{n+1} :

$$f\left(x_{n+1}\right)=f\left(x_n\right)+\left(x_{n+1}-x_n\right)f'\left(x_n\right)=0$$

so that:

$$x_{n+1}=x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)}$$

This is therefore a fixed point method

$$g(x)=x-\frac{f(x)}{f'(x)}$$

One gets:

$$g'(x)=\frac{f(x)f''(x)}{f'(x)}$$

and

$$g''(x)=\frac{f''(x)}{f'(x)}-2\frac{f(x)f''(x)^2}{f'(x)^3}+\frac{f(x)f'''(x)}{f'(x)^2}$$

Note that !!!

$$g'(s)=0$$

The Newton method is a second order method

In addition there always exist an interval around s such that $|g'(s)| < 1$. The Newton method always converges provided the start value x_0 is close enough to the solution.

Application to the quasi Newton method

In that case, one gets:

$$f\left(x_{n+1}\right) = f\left(x_n\right) + \left(x_{n+1} - x_n\right) K = 0$$

where K is constant. Therefore

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{K}$$

and

$$g(x) = x - \frac{f(x)}{K}$$

and

$$g'(x) = 1 - \frac{f'(x)}{K}$$

As $g'(s) \neq 0$, this method is a first order method. It converges for K such that:

$$-1 < 1 - \frac{f'(s)}{K} < 1$$

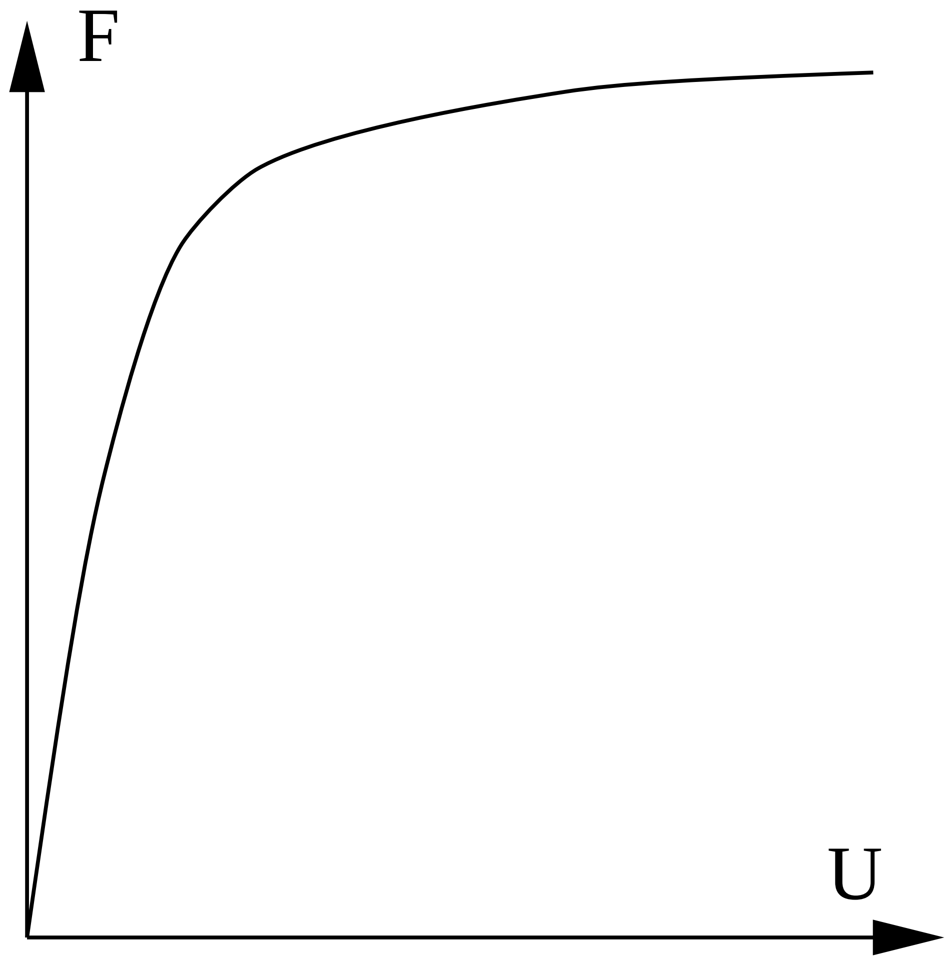
Riks method
Control

The “natural” problem control mode is to impose the external forces $\{F_e\}$. This control works if the load increases with displacement.

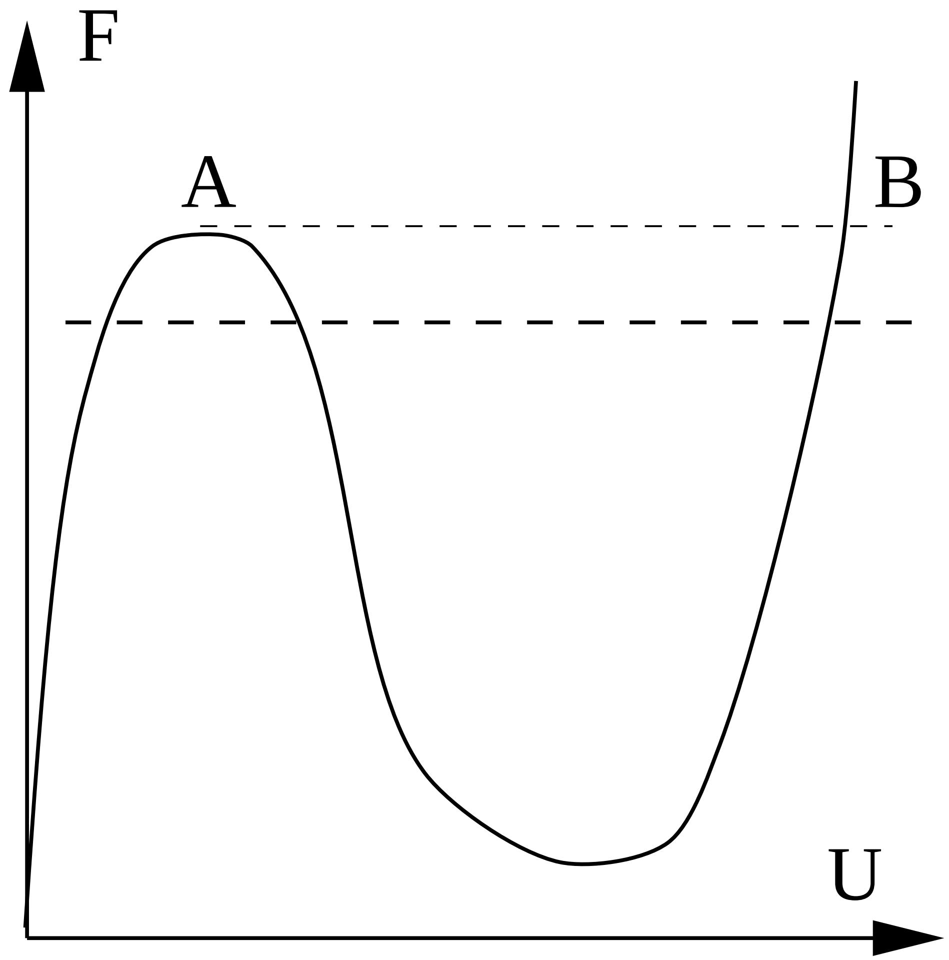
In the case of a limit load, a displacement control is needed.

In the case of snap-back instabilities, a mixte load—a displacement control is needed.

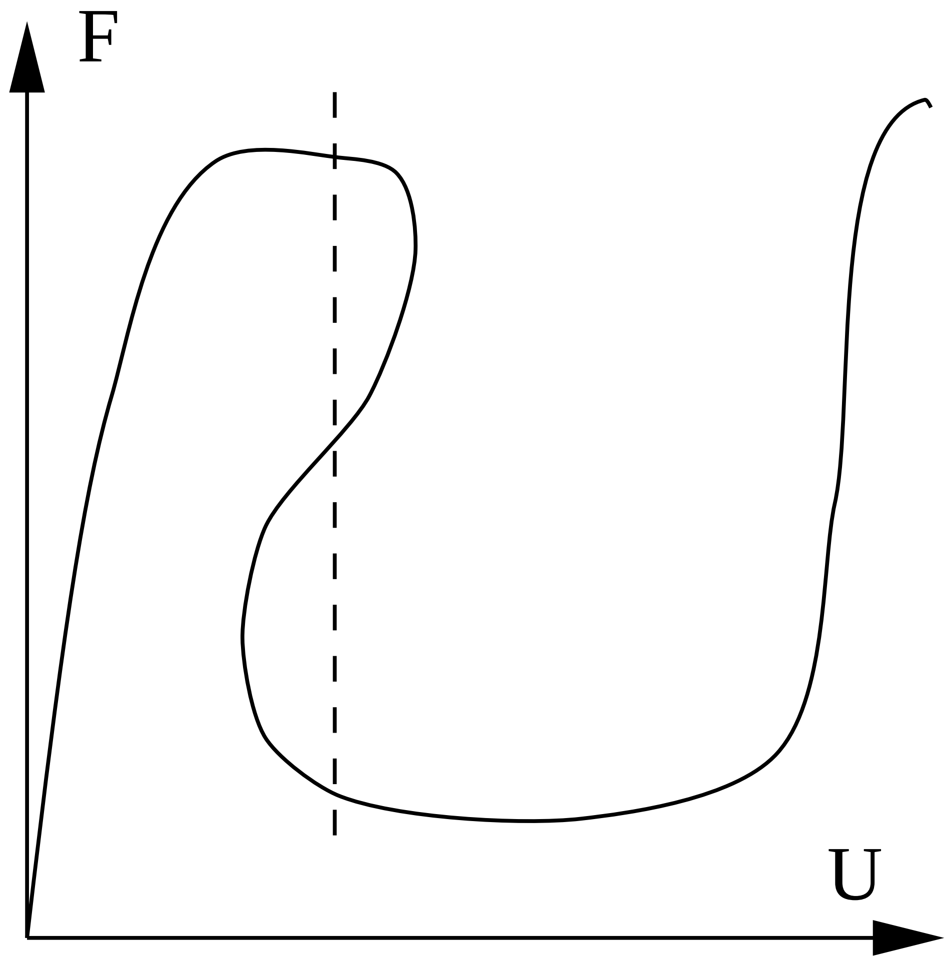
No problem



Problem under load control



Problem under both load and displacement control



Convergence

The convergence of the iterative resolution can be tested according to different methods:

- As the search solution verifies: $\{R\} = \{0\}$, the iterative process is stopped when $\{R\}$ is small enough:

$$||\{R\}||_n < R_\epsilon$$

where R_ϵ is the requested precision. With

$$||\{R\}||_n = \left(\sum_i R_i^n\right)^{\frac{1}{n}}$$

The "inf." norm is often used:

$$||\{R\}||_\infty = \max_i |R_i|$$

- In many cases, the equation $\{R\} = \{0\}$ can be written as: $\{R\}_i - \{R\}_e = \{0\}$ where $\{R\}_e$ is prescribed. A relative error can then be defined:

$$\frac{||\{R\}_i - \{R\}_e||}{||\{R\}_e||} < r_\epsilon$$

where r_ϵ is the requested precision. Note that in some cases (residual stresses during cooling) $\{R\}_e = \{0\}$ so that a relative error cannot be defined.

- The search can be stopped when the approximate solution is stable, i.e.

$$\left|\left|\{U\}_{k+1} - \{U\}_k\right|\right|_n < U_\epsilon$$

This is not a strict convergence criterion For instance the serie: $x_n = \log n$ verifies the criterion ($x_{n+1} - x_n = \log((n+1)/n)$) ibut does not converge !

Incompressibility / Quasi incompressibility

Some materials are incompressible or quasi–incompressible (rubber, metals during forming, etc. . .).

The incompressibility condition is written as: La condition d'incompressibilité se traduit par la condition

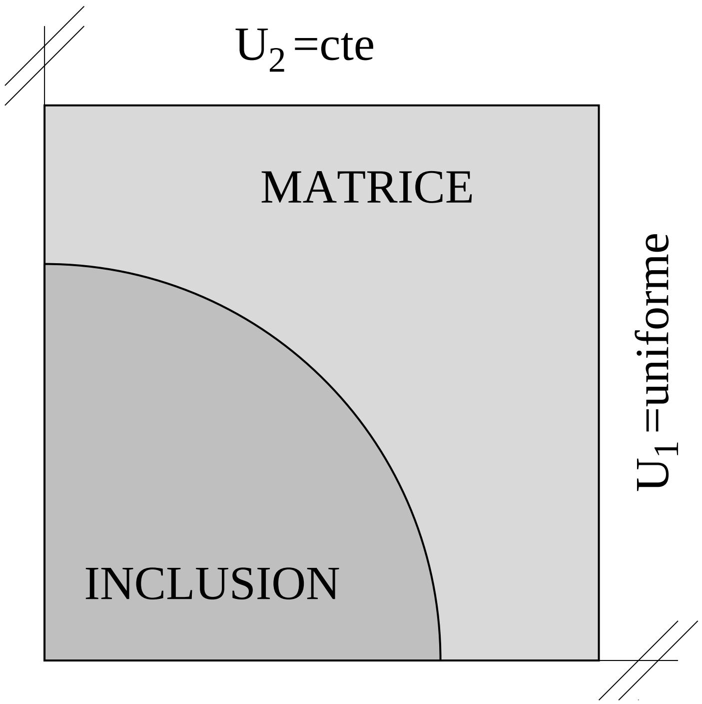
$$\mathrm{div}\,(\underline{\dot{\mathbf{u}}}) = 0$$

A FE formulation based on displacement only does not "naturally" enforce this condition. An enriched method must be used.

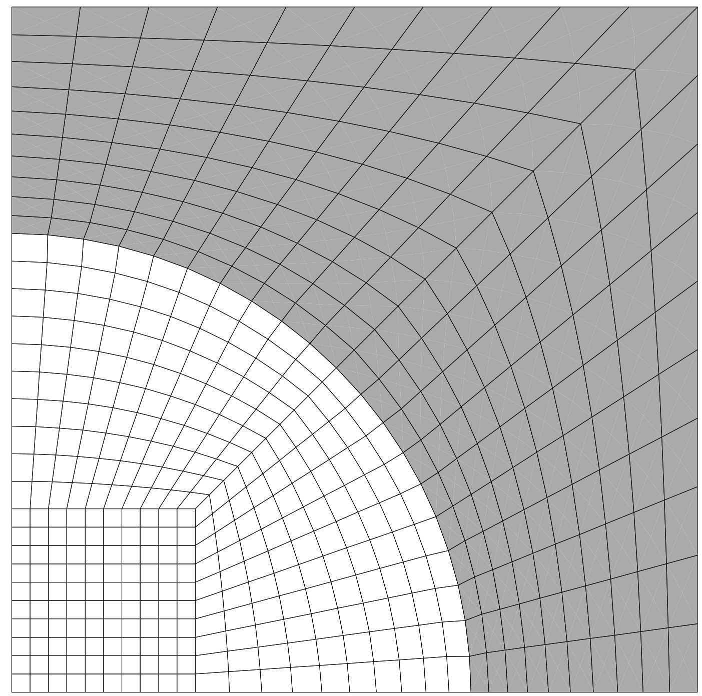
The stress tensor can be separated into a deviatoric component $\underset{\sim}{\mathbf{s}}$ and an hydrostatic component p so that: que l'on a :

$$\underset{\sim}{\sigma} = \underset{\sim}{\mathbf{s}} - p\underset{\sim}{\mathbf{1}}$$

Symptomes — 1



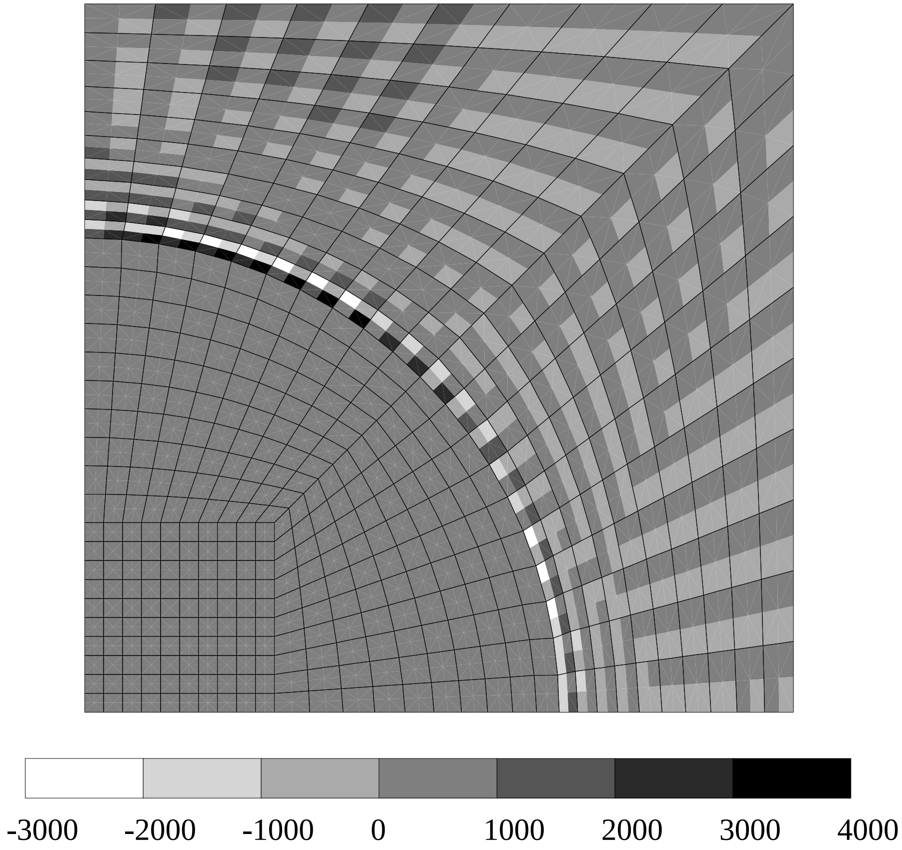
boundary conditions



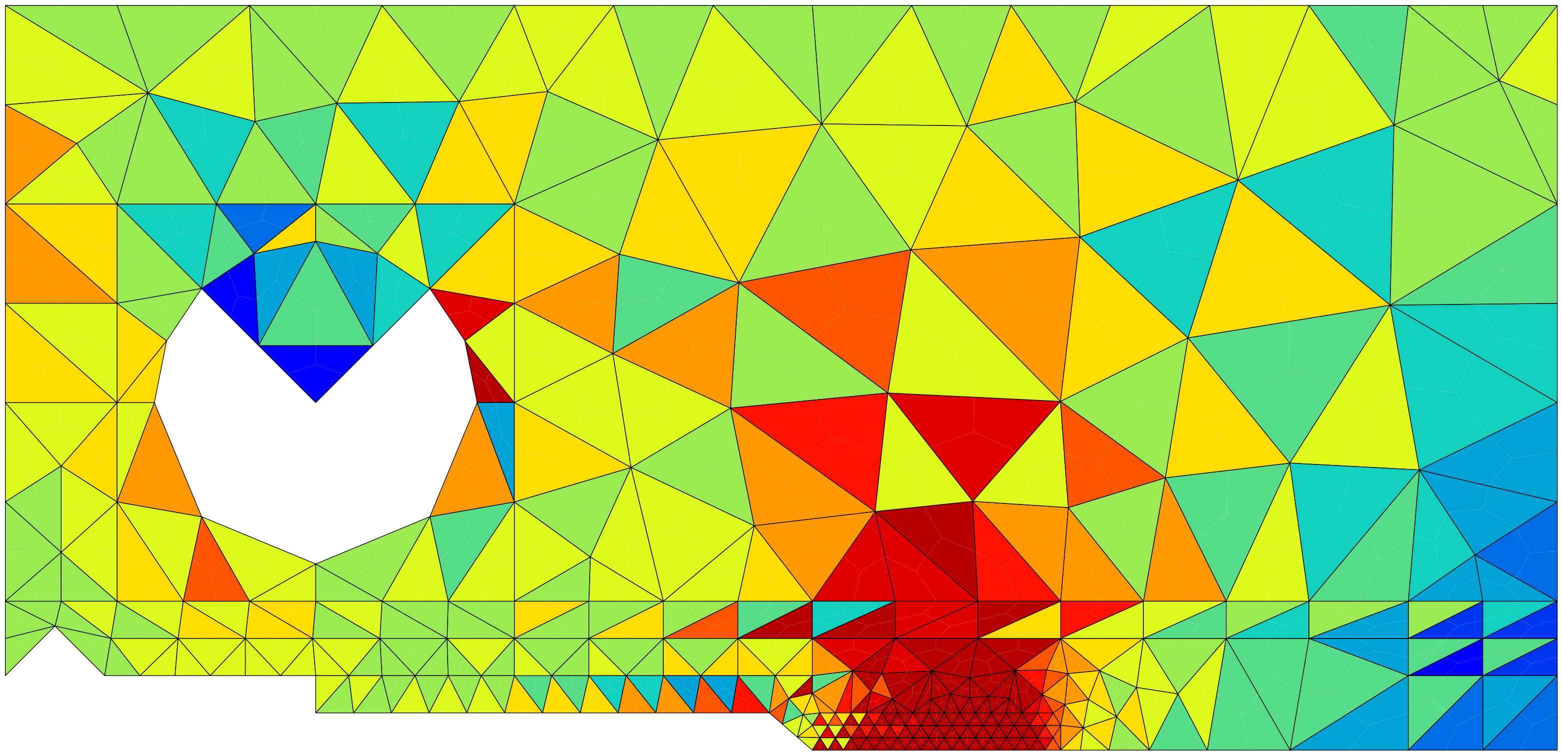
mesh

Inclusion	Young's modulus 400 GPa, Poisson coefficient 0.2
Matrix	Young's modulus 70 GPa, Poisson coefficient 0.3, yield stress 200 MPa

- Result (pressure)



Symptomes — 2



Linear (3 nodes) triangles

Solution 1

- The first solution consists in post-processing the data in order to average the pressure within each element:

$$\bar{p} = \frac{1}{V}\int_V p dV$$

- A new stress field is build:

$$\underset{\sim}{\sigma} = \underset{\sim}{\mathbf{s}} - p\underset{\sim}{\mathbf{1}} \quad \rightarrow \quad \underset{\sim}{\sigma}^* = \underset{\sim}{\mathbf{s}} - \bar{p}\underset{\sim}{\mathbf{1}}$$

- This solution can be useful but is not general (does not work for T3)

Approximated formulation : selective integration

One uses a selective integration of the volume variation.

- The strain tensor is related to the nodal displacement by

$$\underset{\sim}{\varepsilon} = [B] \,.\, \{U\}$$

- $[B]$ can be separated into a deviatoric part $[B_{dev}]$ and a dilatation part $[B_{dil}]$:

$$[B] = [B_{dev}] + [B_{dil}]$$

- $[B_{dil}]$ is then avaraged over the element:

$$\left[\bar{B}_{dil}\right] = \frac{1}{|V_e|} \int_{V_e} [B_{dil}] \, dV$$

- A modified $[B]$ is reconstructed;

$$[B^*] = [B_{dev}] + \left[\bar{B}_{dil}\right]$$

- Deformation is then computed using $[B^*]$:

$$\underset{\sim}{\varepsilon} = [B^*] \,.\, \{U\}$$

- The volume variation is therefore constant in the element
- Once again the method cannot be applied to linear triangles and tetrahedrons. It can be applied to quadraric triangles and tetrahedrons

Approximated Formulation : penalisation

- In that case the material behavior is incompressible; this implies that only $\underset{\sim}{\mathbf{s}}$ and not $\underset{\sim}{\sigma}$ can be obtained from the material. • pressure is computed for each element

$$p = -\kappa u_{i,i}$$

- κ : numerical penalisation factor (compressibility)
- $\underset{\sim}{\sigma} = \underset{\sim}{\mathbf{s}} - p\underset{\sim}{\mathbf{1}}$
- Ifκ is large enough : $\mathrm{div}\underline{\mathbf{u}} \approx 0$
- Unknowns: displacements $\{U\}$.
- Internes forces are still given by:

$$\int_{V_e} [B]^T . \underset{\sim}{\sigma} dV$$

- the elementary stiffness matrix is now given by:

$$[K_e] = \int_{V_e} \left([B]^T . \underset{\approx}{\mathbf{L}} . [B] + \lambda([B]^T . [m]^T) \otimes ([m] . [B]) \right) dV$$

- $[m] = [1\,1\,1\,0\,0\,0]$

Mixted pressure–displacement formualtion

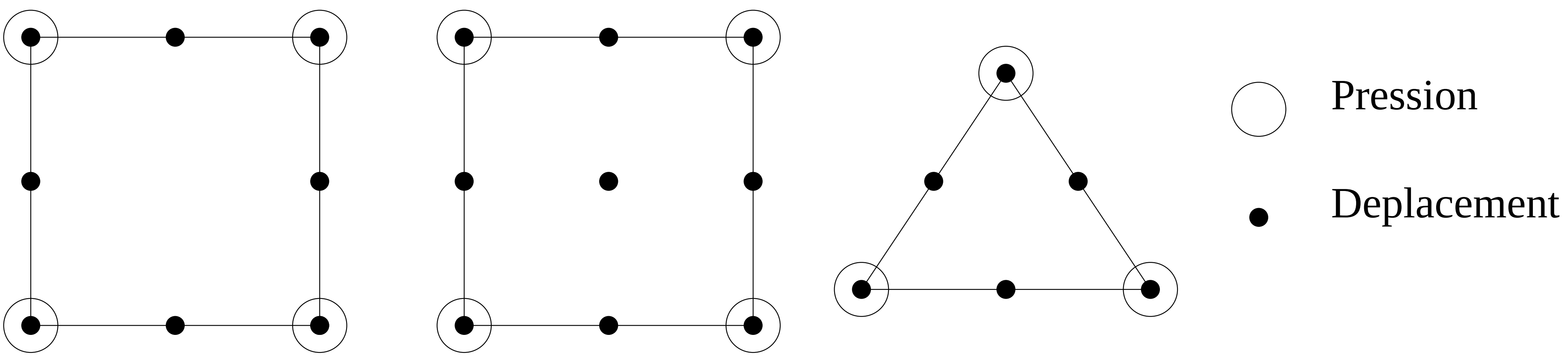
New degrees of freedom are added to represent the pressure field which is defined as:

$$p = N^{k'} p^{k'}$$

Displacement and positions are given by:

$$\underline{\mathbf{u}} = N^k \underline{\mathbf{u}}^k \qquad \underline{\mathbf{x}} = N^{k''} \underline{\mathbf{x}}^{k''}$$

A higher order interpolation is used for the displacement than for the pressure so that strains (derivative of the displacement) are the same order than the pressure.



- For linear element with respect to $\vec{u}$, the pressure is constant in each element

The mechanical equilibrium and the incompressibility condition must be solved simultaneously

$$\underset{\sim}{\sigma} = \underset{\sim}{\mathbf{s}} - p\underset{\sim}{\mathbf{1}}$$

$$\Delta\underset{\sim}{\mathbf{s}} = \underset{\approx}{\mathbf{L}} : \Delta\underset{\sim}{\varepsilon} \qquad \text{material}$$

$$p = [H_p]\,\{p\} \qquad \text{element}$$

$$\Delta\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{L}}.\,[B]\,.\,\{\Delta u\} - ([H_p]\,.\,\{\Delta p\})\,[m]^T$$

$$[m] = [1\,1\,1\,0\,0\,0]$$

$$[H_p] = \left[N_1 \;\ldots\; N_{n_p}\right]$$

PVW

- PVW

$$W_i = \int_V \left(\sigma : \dot{\underset{\sim}{\varepsilon}} + p \,\mathrm{div} u \right) dV$$

- Internal forces

$$\{I\} = \left\{ \begin{array}{c} \{u\} \\ \{p\} \end{array} \right\} \qquad \{F_i\} = \left\{ \begin{array}{c} \int_{V_e} [B]^T . \underset{\sim}{\sigma} dV \\ \int_{V_e} [H_p]^T \,\mathrm{div}\underline{\mathbf{u}} dV \end{array} \right\}$$

- Stiffness matrix

$$[K_e] = \left[\begin{array}{cc} \int_{V_e} [B]^T . \underset{\approx}{\mathbf{L}} . [B] \, dV & \int_{V_e} ([B]^T . [m]) \otimes [H_p] \, dV \\ \int_{V_e} [H_p]^T \otimes ([m] . [B]) dV & [0] \end{array} \right]$$