

Mécanique des milieux continus

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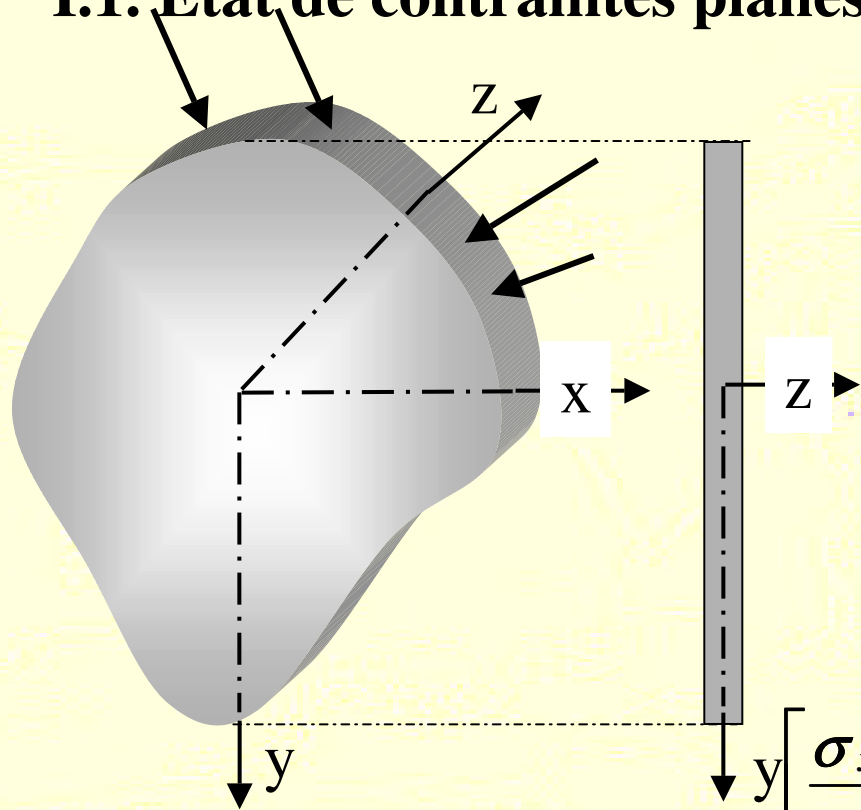
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CHAPITRE VII

élasticité plane

I. Les deux états de l'élasticité plane

I.1. Etat de contraintes planes

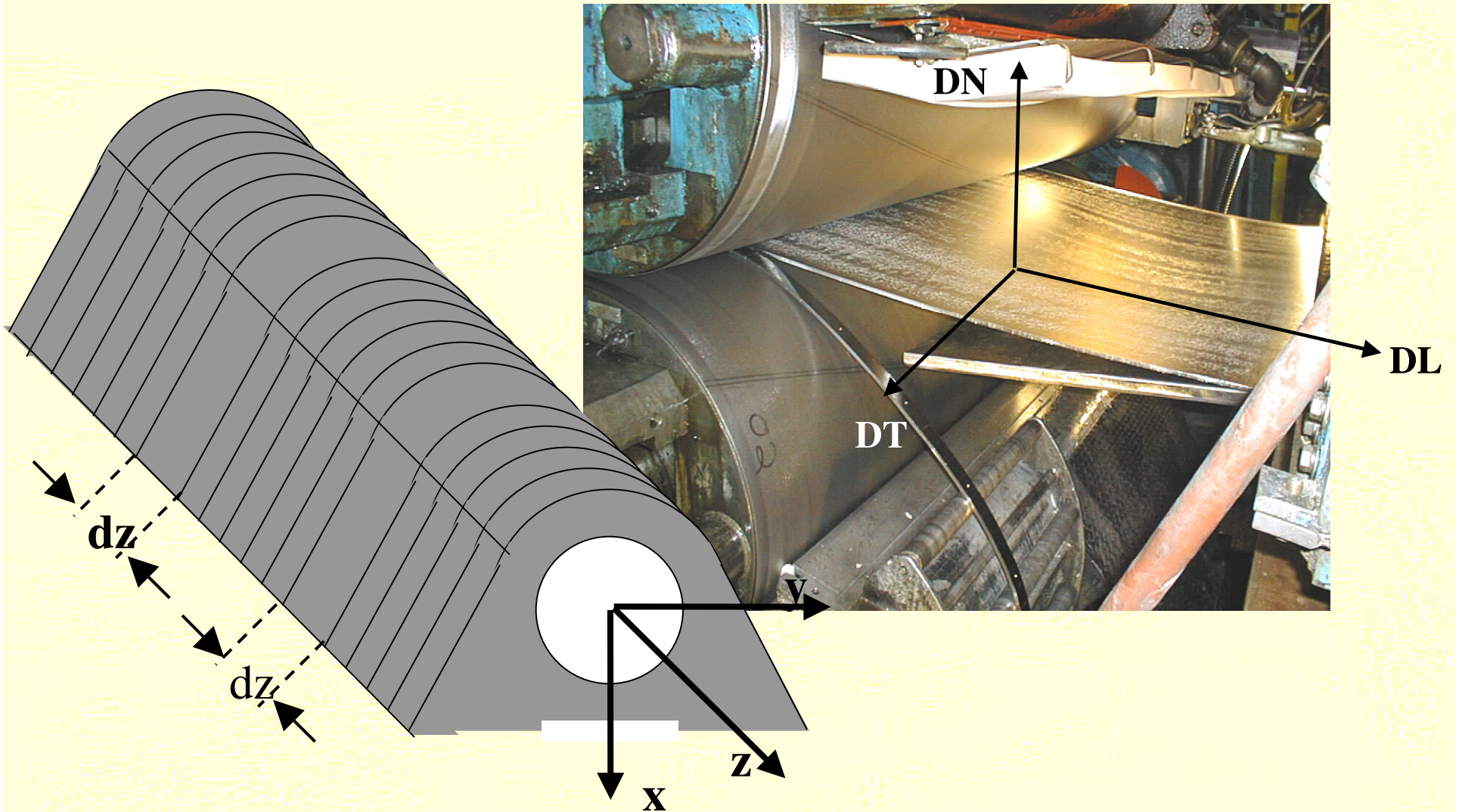


$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \frac{\sigma_{xx} - \nu \sigma_{yy}}{E} & \frac{\sigma_{xy}}{2G} & 0 \\ \frac{\sigma_{xy}}{2G} & \frac{\sigma_{yy} - \nu \sigma_{xx}}{E} & 0 \\ 0 & 0 & -\frac{\nu(\sigma_{xx} + \sigma_{yy})}{E} \end{bmatrix}$$

I. Les deux états de l'élasticité plane

I.2. Etat de déformations planes



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I.2. Etat de déformations planes

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \boldsymbol{\varepsilon}_{xy} & \mathbf{0} \\ \boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}_{zz} = \frac{1}{E} \left[\boldsymbol{\sigma}_{zz} - \nu (\boldsymbol{\sigma}_{xx} + \boldsymbol{\sigma}_{yy}) \right] = \mathbf{0} \Rightarrow \boldsymbol{\sigma}_{zz} = \nu (\boldsymbol{\sigma}_{xx} + \boldsymbol{\sigma}_{yy})$$

$$\left\{ \begin{array}{l} \boldsymbol{\sigma}_{xx} = \frac{E}{(1+\nu)(1-2\nu)} \left[(1-\nu)\boldsymbol{\varepsilon}_{xx} + \nu\boldsymbol{\varepsilon}_{yy} \right] \\ \boldsymbol{\sigma}_{yy} = \frac{E}{(1+\nu)(1-2\nu)} \left[\nu\boldsymbol{\varepsilon}_{xx} + (1-\nu)\boldsymbol{\varepsilon}_{yy} \right] \\ \boldsymbol{\sigma}_{zz} = \nu (\boldsymbol{\sigma}_{xx} + \boldsymbol{\sigma}_{yy}) \end{array} \right.$$

I. Les deux états de l'élasticité plane

I.2. Etat de déformations planes

Contraintes planes



déformations planes

$$E \rightarrow \frac{E}{(1-\nu^2)} \quad \nu \rightarrow \frac{\nu}{(1-\nu)}$$

Déformations planes



Contraintes planes

$$E \rightarrow \frac{E(1+2\nu)}{(1+\nu)^2} \quad \nu \rightarrow \frac{\nu}{(1+\nu)}$$

II. Les équations fondamentales de l'élasticité plane

II.1. Equations de compatibilité

$$\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} \varepsilon_{ij} + \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \varepsilon_{kk} - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \varepsilon_{ik} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \varepsilon_{jk} = 0$$



$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0$$

III. La fonction d'Airy

III.1. Forces de volume nulles

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \end{cases}$$

$$\left\{ \begin{array}{l} \sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} \\ \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} \\ \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \end{array} \right.$$

III. La fonction d'Airy

III.1. Forces de volume nulles

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) = \frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

III. La fonction d'Airy

III.2. Forces dérivant d'un potentiel

$$\vec{F} = \overrightarrow{\text{grad}V} = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{bmatrix}$$

$$\begin{cases} \frac{\partial(\sigma_{xx} + V)}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0 \\ \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial(\sigma_{yy} + V)}{\partial y} = 0 \end{cases}$$

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$$\sigma_{xx} + V = \frac{\partial^2\Phi}{\partial y^2}$$

$$\sigma_{yy} + V = \frac{\partial^2\Phi}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}$$

III. La fonction d'Airy

III.2. Forces dérivant d'un potentiel

$$\left\{ \begin{array}{l} \sigma_{xx} + V = \frac{\partial^2 \Phi}{\partial y^2} \\ \sigma_{yy} + V = \frac{\partial^2 \Phi}{\partial x^2} \\ \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \end{array} \right.$$

Déformations planes

$$\Delta \Delta \Phi = \frac{1-2\nu}{1-\nu} \Delta V$$

Contraintes planes

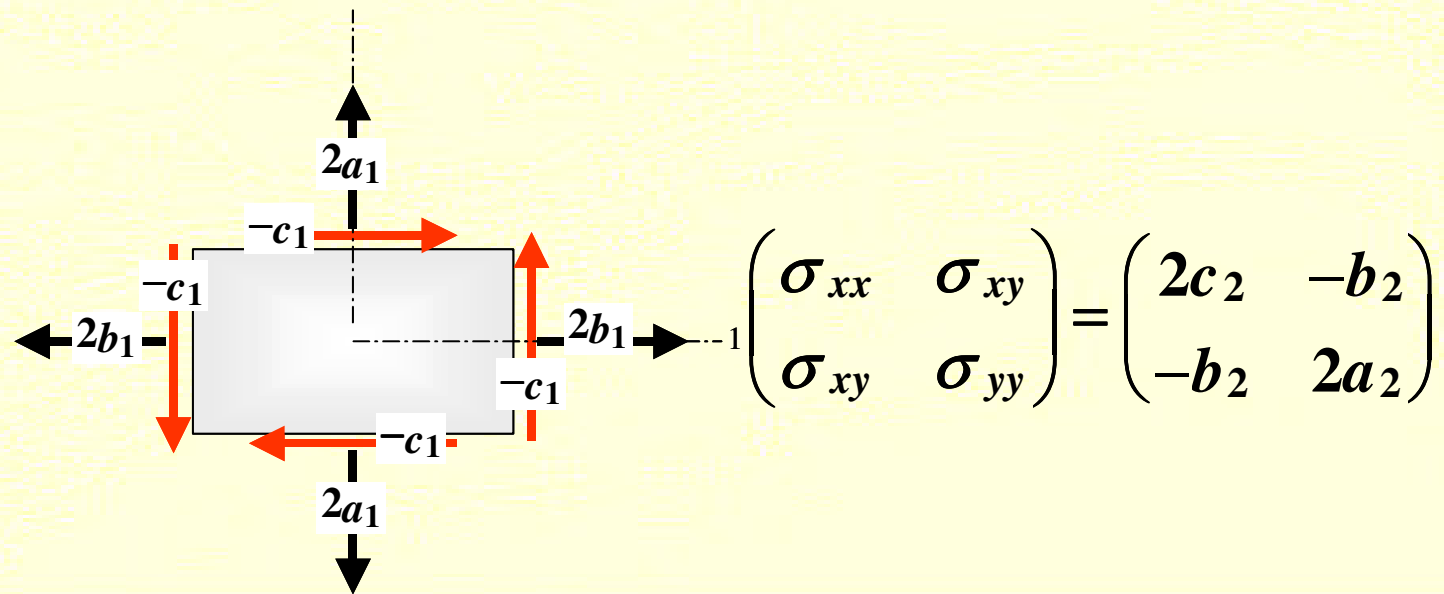
$$\Delta \Delta \Phi = (1-\nu) \Delta V$$

IV. Application à la théorie des poutres

IV.1. Solution de l'équation biharmonique par des polynômes

$$\frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

Polynôme biquadratique $\varphi_2(x, y) = a_2 x^2 + b_2 xy + c_2 y^2$



IV. Application à la théorie des poutres

IV.1. Solution de l'équation biharmonique par des polynômes

$$\frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

Polynôme bicubique $\varphi_3(x, y) = a_3 x^3 + b_3 x^2 y + c_3 x y^2 + d_3 y^3$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} 2c_3 x + 6d_3 y & -2b_3 x - 2c_3 y \\ -2b_3 x - 2c_3 y & 6a_3 x + 2b_3 y \end{pmatrix}$$

IV. Application à la théorie des poutres

IV.2. Détermination des déplacements

Contraintes planes

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \frac{\sigma_{xx} - \nu\sigma_{yy}}{E} & \frac{\sigma_{xy}}{2G} & 0 \\ \frac{\sigma_{xy}}{2G} & \frac{\sigma_{yy} - \nu\sigma_{xx}}{E} & 0 \\ 0 & 0 & -\frac{\nu(\sigma_{xx} + \sigma_{yy})}{E} \end{bmatrix}$$

Déformations planes

$$\begin{cases} \varepsilon_{xx} = \frac{(1+\nu)}{E} [(1-\nu)\sigma_{xx} - \nu\sigma_{yy}] \\ \varepsilon_{yy} = \frac{(1+\nu)}{E} [-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}] \end{cases}$$

IV. Application à la théorie des poutres

IV.2. Détermination des déplacements

$$\left. \begin{aligned} \varepsilon_{xx} &= \frac{\partial u(x, y)}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v(x, y)}{\partial y} \end{aligned} \right\} \begin{cases} u(x, y) = \int \varepsilon_{xx} dx + f_1(y) \\ v(x, y) = \int \varepsilon_{yy} dy + f_2(x) \end{cases}$$

$$\varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial u(x, y)}{\partial y} + \frac{\partial v(x, y)}{\partial x} \right)$$

IV. Application à la théorie des poutres

IV.2. Détermination des déplacements

$$\left\{ \begin{array}{l} u(x, y) = \int \varepsilon_{xx} dx + f_1(y) \\ v(x, y) = \int \varepsilon_{yy} dy + f_2(x) \end{array} \right. \quad \varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial u(x, y)}{\partial y} + \frac{\partial v(x, y)}{\partial x} \right)$$



$$u_1 = a + by \quad v_1 = c - bx$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{xy} = 0$$