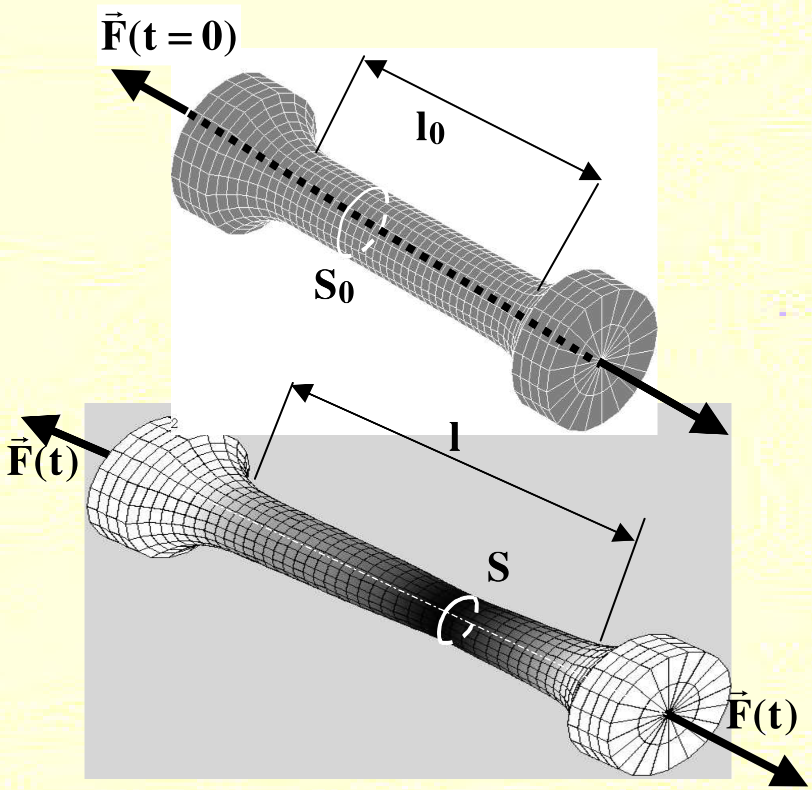
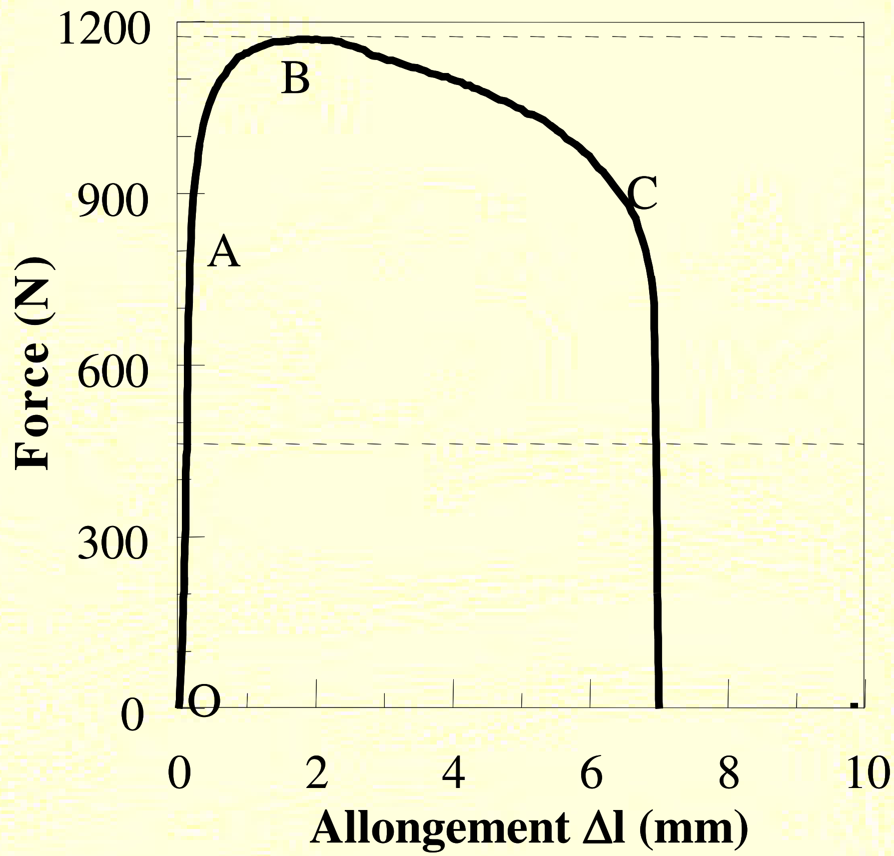
# Chapitre IV : Matériaux élastiques

# I. L'essai de traction

## I.1. Courbe force-allongement

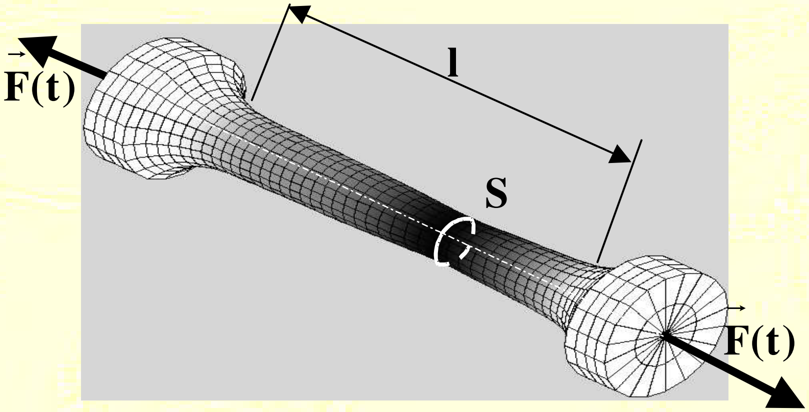


# I. L'essai de traction

## I.2. Courbe contrainte-déformation

### I.2.1. Tenseur déformation et tenseur des contraintes

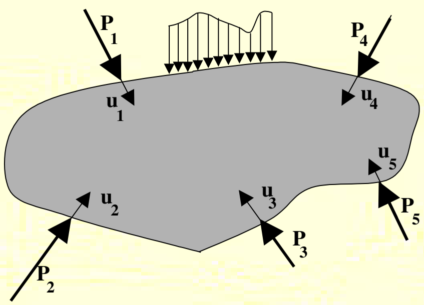


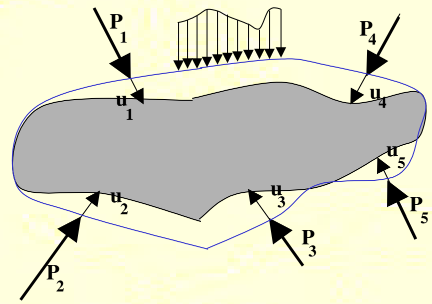

$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F/S \end{bmatrix}$$

$$\vec{x} = \vec{\Phi}\left(\vec{X}, t\right) = \begin{cases} X_1(1-\beta t) \\ X_2(1-\beta t) \\ X_3(1+\alpha t) \end{cases}$$

Langrangien — Eulérien

$$\vec{v} = \frac{\partial \vec{x}}{\partial t} = \begin{cases} -\beta X_1 & = -\beta \dfrac{x_1}{(1-\beta t)} \\ -\beta X_2 & = -\beta \dfrac{x_2}{(1-\beta t)} \\ \alpha X_3 & = \alpha \dfrac{x_1}{(1+\alpha t)} \end{cases}$$



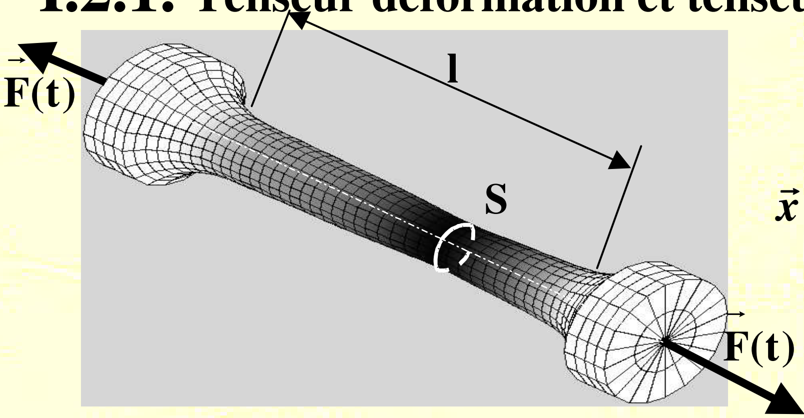




$$dW = \int_V \underbrace{\sigma_{ij}}_{\text{eulérien}} \underbrace{\Delta\varepsilon_{ij}}_{\text{eulérien}} \, dv \qquad \dot{W} = \int_V \sigma_{ij}\dot{\varepsilon}_{ij} dv$$

# I. L'essai de traction

## I.2. Courbe contrainte-déformation

### I.2.1. Tenseur déformation et tenseur des contraintes




$$\vec{x} = \vec{\Phi}\left(\vec{X}, t\right) = \begin{cases} X_1(1-\beta t) \\ X_2(1-\beta t) \\ X_3(1+\alpha t) \end{cases}$$

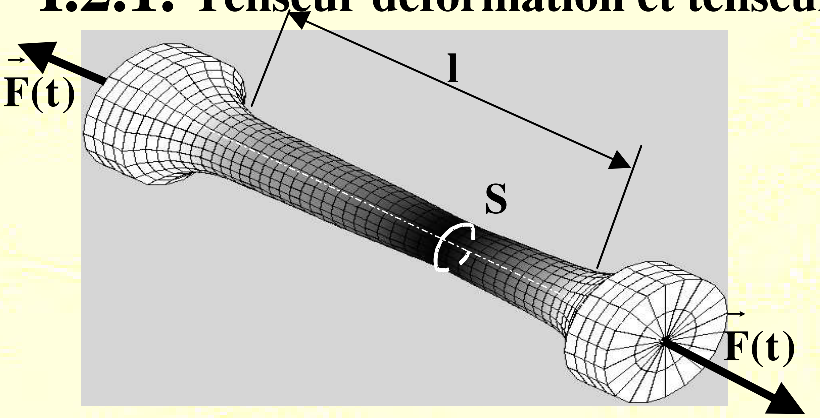
$$\vec{v} = \frac{\partial \vec{x}}{\partial t} = \begin{cases} -\beta X_1 & = -\beta \dfrac{x_1}{(1-\beta t)} \\ -\beta X_2 & = -\beta \dfrac{x_2}{(1-\beta t)} \\ \alpha X_3 & = \alpha \dfrac{x_1}{(1+\alpha t)} \end{cases}$$

$$\underline{\underline{\dot{\varepsilon}}} = \begin{bmatrix} \dfrac{-\beta}{(1-\beta t)} & 0 & 0 \\ 0 & \dfrac{-\beta}{(1-\beta t)} & 0 \\ 0 & 0 & \dfrac{\alpha}{(1+\alpha t)} \end{bmatrix}$$

# I. L’essai de traction

## I.2. Courbe contrainte-déformation

### I.2.1. Tenseur déformation et tenseur des contraintes



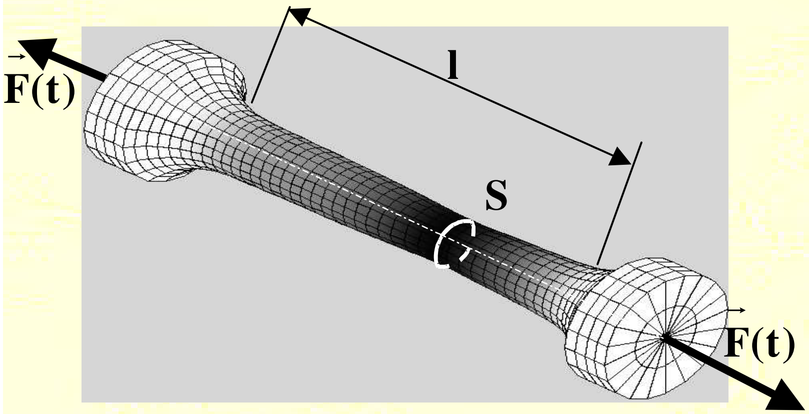

$$\dot{\underline{\underline{\varepsilon}}} = \begin{bmatrix} \frac{-\beta}{(1-\beta t)} & 0 & 0 \\ 0 & \frac{-\beta}{(1-\beta t)} & 0 \\ 0 & 0 & \frac{\alpha}{(1+\alpha t)} \end{bmatrix}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \ln(1-\beta t) & 0 & 0 \\ 0 & \ln(1-\beta t) & 0 \\ 0 & 0 & \ln(1+\alpha t) \end{bmatrix} \qquad \underline{\underline{\sigma}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F/S \end{bmatrix}$$

# I. L'essai de traction

## I.2. Courbe contrainte-déformation

### I.2.1. Tenseur déformation et tenseur des contraintes

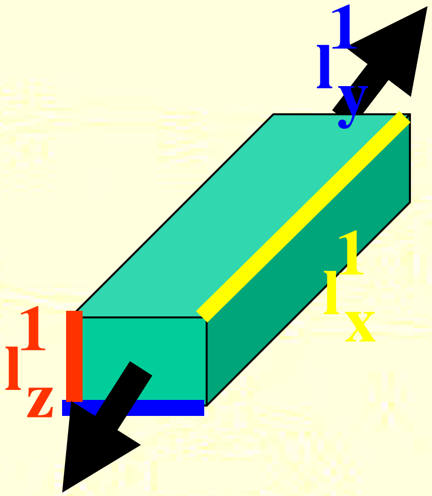


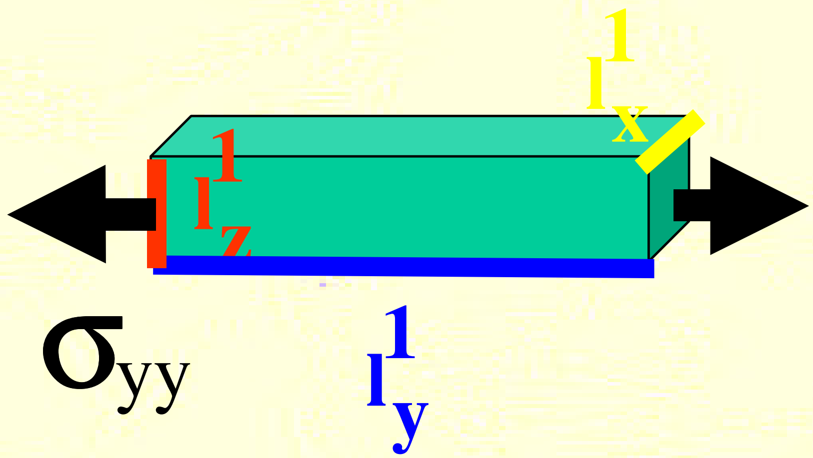

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \ln(1-\overbrace{\beta t}^{inconnu}) & 0 & 0 \\ 0 & \ln(1-\beta t) & 0 \\ 0 & 0 & \ln(1+\underbrace{\alpha t}_{imposé}) \end{bmatrix}$$

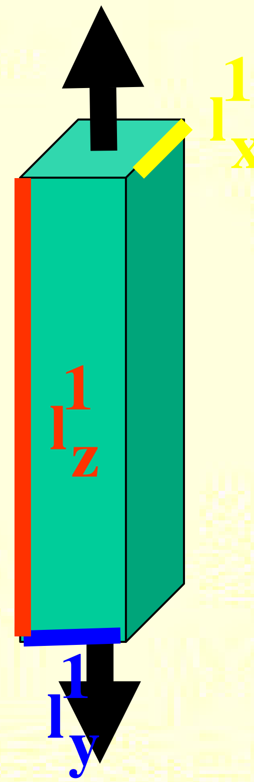
$$\left.\begin{aligned} X_3 &= l_0 \\ x_3 &= l = l_0 + \Delta l \\ x_3 &= l = l_0 + \underbrace{\alpha t}_{imposé} \end{aligned}\right\} \rightarrow \varepsilon_{33} = \ln(1+\frac{\Delta l}{l_0})$$

$$\beta = f(\alpha)? \rightarrow \begin{cases} comportement \\ des \\ matériaux \end{cases}$$

## II. Matériaux élastiques linéaire isotropes
## II. 1. Traction uniaxiale




$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$$

$$\varepsilon_{yy} = -\nu\varepsilon_{xx} = -\nu\frac{\sigma_{xx}}{E}$$

$$\varepsilon_{zz} = -\nu\varepsilon_{xx} = -\nu\frac{\sigma_{xx}}{E}$$

$$\varepsilon_{xx} = -\nu\varepsilon_{yy} = -\nu\frac{\sigma_{yy}}{E}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E}$$

$$\varepsilon_{zz} = -\nu\varepsilon_{yy} = -\nu\frac{\sigma_{yy}}{E}$$

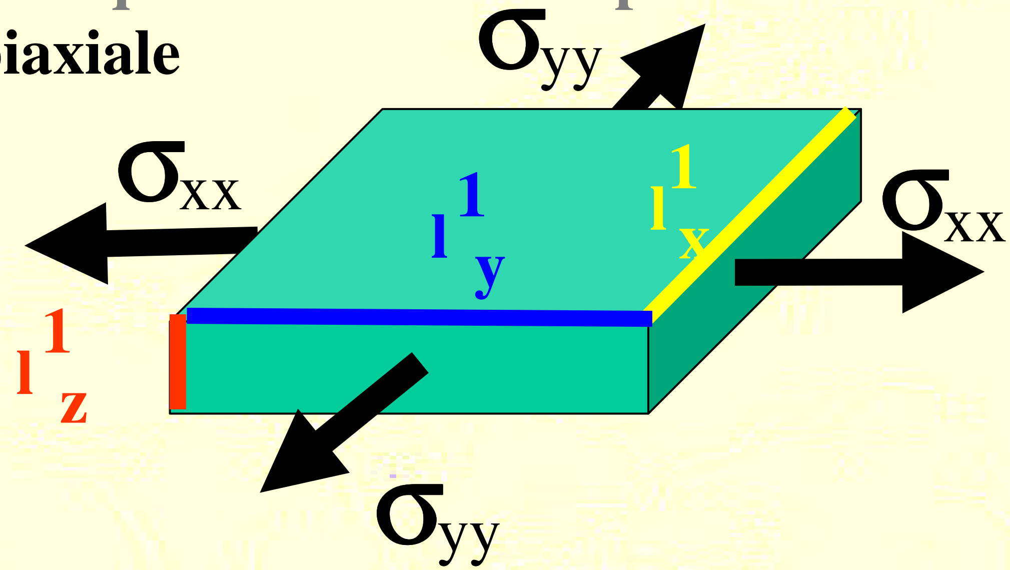
$$\varepsilon_{xx} = -\nu\varepsilon_{zz}$$

$$\varepsilon_{yy} = -\nu\varepsilon_{zz}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E}$$

# II. Matériaux élastiques linéaire isotropes

## II. 2. Traction biaxiale




| Traction selon x | Traction selon y | Traction selon x et y |
|---|---|---|
| $\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$ | $\varepsilon_{xx} = -\nu \frac{\sigma_{yy}}{E}$ | $\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E}$ |
| $\varepsilon_{yy} = -\nu \frac{\sigma_{xx}}{E}$ | $\varepsilon_{yy} = \frac{\sigma_{yy}}{E}$ | $\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{xx}}{E}$ |
| $\varepsilon_{zz} = -\nu \frac{\sigma_{xx}}{E}$ | $\varepsilon_{zz} = -\nu \frac{\sigma_{yy}}{E}$ | $\varepsilon_{zz} = -\nu \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E}$ |

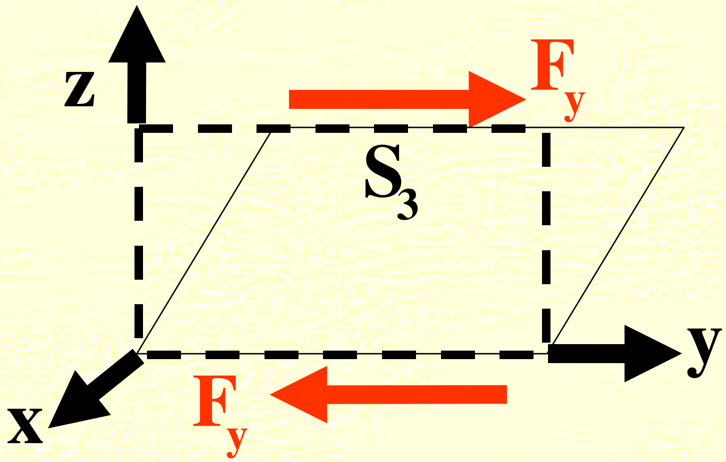
## II. 3. Traction le long de trois axes orthogonaux

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E}$$

## II. 4. Sollicitation en cisaillement




$$\varepsilon_{xy} = \frac{\sigma_{xy}}{2G}$$

$$\varepsilon_{xz} = \frac{\sigma_{xz}}{2G}$$

$$\varepsilon_{yz} = \frac{\sigma_{yz}}{2G}$$

$$G = \frac{E}{2(1+\nu)}$$

# II. Matériaux élastiques linéaire isotropes

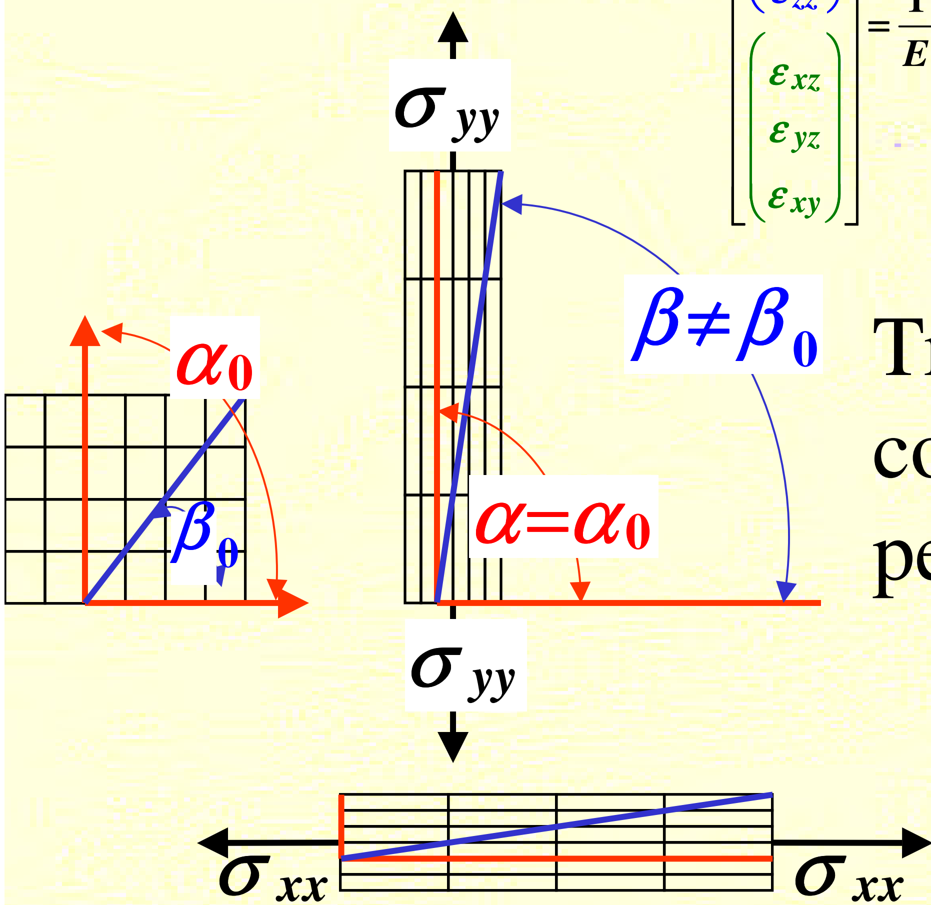
## II. 5. La loi de Hooke complète pour un matériau isotrope

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & 1+\nu & 0 & 0 \\ & & & 0 & 1+\nu & 0 \\ & & & 0 & 0 & 1+\nu \end{bmatrix} * \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{bmatrix}$$

## II. Matériaux élastiques linéaire isotropes

### II. 5. La loi de Hooke complète pour un matériau isotrope

$$\begin{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{pmatrix} \\ \begin{pmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \\ \varepsilon_{xy} \end{pmatrix} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} \begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} 1+\upsilon & 0 & 0 \\ 0 & 1+\upsilon & 0 \\ 0 & 0 & 1+\upsilon \end{pmatrix} \end{bmatrix} * \begin{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix} \\ \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{pmatrix} \end{bmatrix}$$

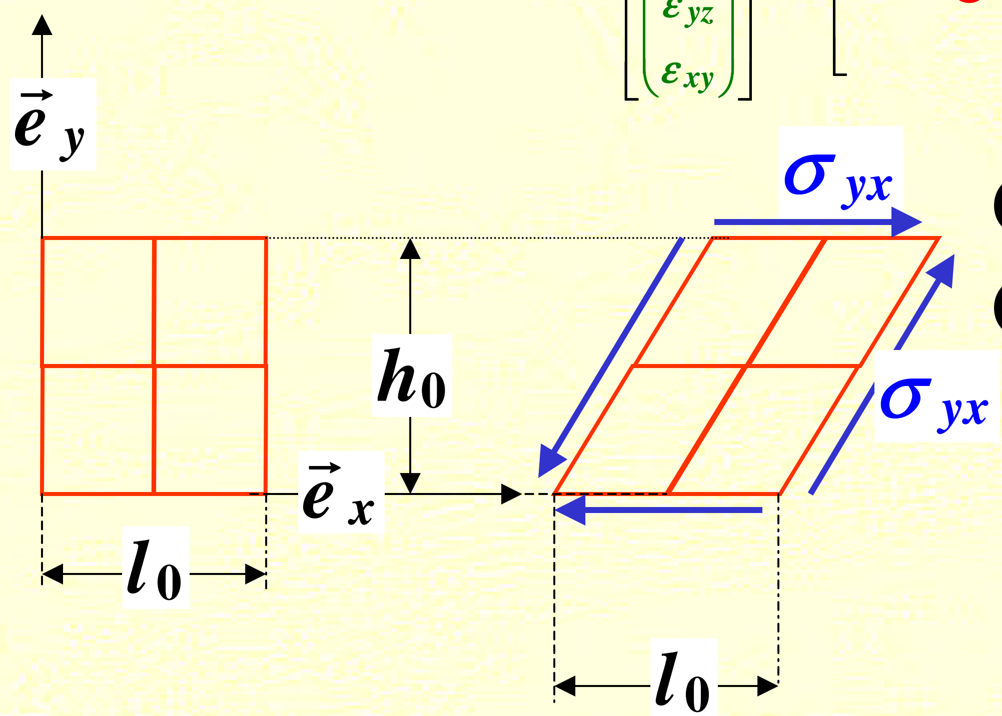



Traction selon x ou y conserve les sections droites perpendiculaires à x et y

## II. Matériaux élastiques linéaire isotropes

# II. 5. La loi de Hooke complète pour un matériau isotrope

$$\begin{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{pmatrix} \\ \begin{pmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \\ \varepsilon_{xy} \end{pmatrix} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} \begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} 1+\upsilon & 0 & 0 \\ 0 & 1+\upsilon & 0 \\ 0 & 0 & 1+\upsilon \end{pmatrix} \end{bmatrix} * \begin{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix} \\ \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{pmatrix} \end{bmatrix}$$




**Cisaillement :**

**Conserve les longueurs**

## II. 5. La loi de Hooke complète pour un matériau isotrope

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} * \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \sqrt{2}\varepsilon_{xz} \\ \sqrt{2}\varepsilon_{yz} \\ \sqrt{2}\varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} * \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sqrt{2}\sigma_{xz} \\ \sqrt{2}\sigma_{yz} \\ \sqrt{2}\sigma_{xy} \end{bmatrix}$$

## II. 5. La loi de Hooke complète pour un matériau isotrope

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \sqrt{2}\varepsilon_{xz} \\ \sqrt{2}\varepsilon_{yz} \\ \sqrt{2}\varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} * \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sqrt{2}\sigma_{xz} \\ \sqrt{2}\sigma_{yz} \\ \sqrt{2}\sigma_{xy} \end{bmatrix}$$

$$\vec{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sqrt{2}\sigma_{xz} \\ \sqrt{2}\sigma_{yz} \\ \sqrt{2}\sigma_{xy} \end{bmatrix} \quad \vec{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \sqrt{2}\varepsilon_{xz} \\ \sqrt{2}\varepsilon_{yz} \\ \sqrt{2}\varepsilon_{xy} \end{bmatrix} \quad \underline{\underline{M}} = \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix}$$

## II. 5. La loi de Hooke complète pour un matériau isotrope

$$\vec{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sqrt{2}\sigma_{xz} \\ \sqrt{2}\sigma_{yz} \\ \sqrt{2}\sigma_{xy} \end{bmatrix} \qquad \underline{\underline{M}} = \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \qquad \vec{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \sqrt{2}\varepsilon_{xz} \\ \sqrt{2}\varepsilon_{yz} \\ \sqrt{2}\varepsilon_{xy} \end{bmatrix}$$

$$\varepsilon_{ij} = M_{ijkl}\sigma_{kl}$$

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl}$$

## III. La loi de Hooke complète pour un matériau anisotrope

$$\left.\begin{array}{l} \sigma_{ij} = L_{ijkl}\varepsilon_{kl} \\ \sigma_{ij} = \sigma_{ji} \\ \varepsilon_{kl} = \varepsilon_{lk} \end{array}\right\} \begin{array}{l} L_{ijkl} = L_{jikl} \\ L_{ijkl} = L_{ijlk} \end{array}$$

# IV. L'énergie de déformation

## IV.1. Expression de l'énergie en fonction des contraintes et des déformations

$$dW = \int_V \sigma_{ij} \Delta\varepsilon_{ij} dv \qquad W = \int_V \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij} dv$$

$$W_{vol} = \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij}$$

$$W_{vol} = \int_0^{\varepsilon} \vec{\sigma}^T d\vec{\varepsilon}$$

# IV. L'énergie de déformation

## IV.1. Expression de l'énergie en fonction des contraintes et des déformations

$$W_{vol} = \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij}$$

$$\varepsilon_{ij} = M_{ijkl}\sigma_{kl}$$

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl}$$

$$\rightarrow W_{vol} = \frac{1}{2}\underline{\underline{\sigma}} : \underline{\underline{\underline{\underline{M}}}} : \underline{\underline{\sigma}}$$

$$= \frac{1}{2}\underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{L}}}} : \underline{\underline{\sigma}}$$

# IV. L'énergie de déformation

## IV.1. Propriétés de M et L

$$W_{vol} = \frac{1}{2}\vec{\sigma}^T M \vec{\sigma} \;\geq 0 \rightarrow M\, est\, défini\; postif$$

$$W_{vol} = \frac{1}{2}\vec{\varepsilon}^T L \vec{\varepsilon} \;\geq 0 \rightarrow L\, est\, défini\; postif$$

# IV. L'énergie de déformation

## IV.1. Propriétés de M et L

$$W_{vol} = \frac{1}{2}\varepsilon_{ij} L_{ijkl} \varepsilon_{kl}$$

$$\downarrow$$

$$L_{ijkl} = \frac{\partial^2 W_{vol}}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \frac{\partial^2 W_{vol}}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} = L_{klij}$$