1 Bending of a beam with a rectangular section

The beam in figure 1 has a rectangular section (thickness 2h, width b). The applied load is a pure bending moment. As a consequence, the internal forces are represented by a uniaxial stress tensor, where the only non-zero component $\sigma_{11}$ linearly depends on $x_3$. The exercise starts with a purely static analysis. A kinematic analysis will follow in question 6.

The behaviour of the material is elastic ($E, \nu$) perfectly plastic ($\sigma_y$).

1. What are the stress and the strain distributions in the elastic domain?

Pure bending around $x_2$ axis of a beam whose neutral axis is $x_1$ is characterized by a onedimensional stress tensor. The component $\sigma_{11}$ is linear wrt $x_3$ (let us assume $\sigma_{11} = kx_3$). As a consequence, the strain component $\varepsilon_{11}$ is also linear wrt $x_3$. On pose $\sigma_{11} = kx_3$. The tensors are represented by:

$$
\begin{pmatrix}
\sigma & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
\sigma/E & 0 & 0 \\
0 & -\nu\sigma/E & 0 \\
0 & 0 & -\nu\sigma/E
\end{pmatrix}
$$

(1)

The stress vector in a current section of normal $e_1$ writes $\sigma_{11}e_1$. This classically produces a zero resulting force on the section ($0 \leq x_2 \leq b$ et $-h \leq x_3 \leq h$). The moment produced by the internal forces can be computed, after denoting by $(0,x_2,x_3)$ the components of $OM$ vector:

$$
M = \iint (OM \times T) dS = M_2 e_2 + M_3 e_3
$$

(2)

with:

$$
M_2 = \iint x_3 \sigma_{11} d x_2 d x_3 = \iint k x_3^2 d x_2 d x_3
$$

(3)

$$
M_3 = - \iint x_2 \sigma_{11} d x_2 d x_3 = - \iint k x_2 x_3 d x_2 d x_3
$$

(4)

The component $M_3$ s equal to zero (integral of $x_3$ between $-h$ and $h$). The expression obtained for $M_2$, that will be simply denoted $M$ later, is (see Fig.2a):

$$
M = kb \int_{-h}^{+h} x_3^2 d x_3 = \frac{2}{3} k b h^3
$$

(6)

One can then express $k$ as a function of the moment, using the notation $I = 2bh^3 / 3$. The value of $\sigma_{11}$ is:

$$
\sigma_{11}(x_3) = \sigma(x_3) = M x_3 / I
$$

(7)

This is an odd function in $x_3$, whose maximum value, $\sigma_m$, reached at $x_3 = h$, is $3M / 2bh^2$. 


2. Which is the value $M_e$ of the applied moment corresponding to the offset of plastic flow?

The condition is reached when $\sigma_m = \sigma_y$, that is: $M_e = 2bh^2\sigma_y/3$

![Figure 2](image)

3. What is the stress distribution when $M$ becomes larger than $M_e$. Show that there is a limit value of the flexure moment $M_\infty$ for which the strain becomes infinite.

The value of the absolute value of $\sigma_{11}$ is limited at $\sigma_y$. Consequently, three zones can be found in the beam width when $M > M_e$. The material remains elastic in the middle of the beam ($-a \leq x_3 \leq a$), and two plastic zones appear, in tension (for $x_3 > a$), and in compression ($x_3 < -a$). In the elastic zone, the stress profile is still linear w.r.t. $x_3$ ($\sigma_{11} = kx_3$). In the plastic zones, the stress values are respectively $\sigma_{11} = +\sigma_y, (x_3 > a)$, and $\sigma_{11} = -\sigma_y, (x_3 < -a)$ (Fig.2b). The two unknowns of the problem, $k$ and $a$, have to verify: – the boundary condition (1):

\[
\int_{-h}^{+h} x_3 \sigma_b dx_3 = M
\]

– the continuity of the stress vector at the internal boundary between elastic and plastic zones: $ka = \sigma_y$, so that $k = \sigma_y/a$.

The value is obtained by replacing $\sigma_{11}$ by its expression in equation (1)

\[
M/2 = \int_0^a x_3(\sigma_y, x_3/a) b dx_3 + \int_a^h x_3 b\sigma_y dx_3
\]

\[
M = b\sigma_y(h^2 - a^2/3)
\]

**Remarks**

– If $a = h$: $M$ value is $M_e = (2/3)b\sigma_yh^2$
– If $a = 0$: $M = M_\infty = b\sigma_yh^2 = 3M_e/2$

In both cases, the elastic and plastic solutions are in good agreement with the limit values.

For $M = M_\infty$, the plastic zone covers the whole beam section. This corresponds to the maximum bending moment. (Fig.2c).

4. What will happen, while releasing external load ($M = 0$) ?

i) after a maximum bending moment $M_m \leq M_e$. 

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ii) after a maximum bending moment $M_e < M_m < M_\infty$?

What is the shape of the so called residual stress field obtained in this last case?

If the maximum value of $M$ during the preloading remains smaller than $M_e$, the whole section remains elastic, so that the beam will recover its initial shape after unloading, without any stress. On the contrary, if the initial loading has produced plastic flow, the local permanent deformation of the material will generate stresses, namely compressive stress in the elongated zone, and tensile stress in the area previously in compression. Assuming that the unloading is fully elastic (it has to be confirmed later) the final state is obtained by means of a superposition procedure. A stress field corresponding to a $-M_m$ moment has to be added to the state obtained at the end of the loading phase. The elastic field is simply defined, for $-h \leq x_3 \leq h$, by:

$$
\sigma_{11} = -\frac{M_m x_3}{I}.
$$

The resulting profile is then (see Fig.3a):

- for $-a \leq x_3 \leq a$ \quad $\sigma = \sigma_y x_3/a - 3M_m x_3/2bh^3$
- for $x_3 \geq a$ \quad $\sigma = \sigma_y - 3M_m x_3/2bh^3$
- for $x_3 \leq -a$ \quad $\sigma = -\sigma_y - 3M_m x_3/2bh^3$

Remarks

- The slope $-3M_m/2bh^3$ is negative for $|x_3| > a$.
- The residual stresses are balanced: $\int_{-h}^{+h} \sigma \, dx_3 = 0$.
- One can check that there is no reverse plastic flow, since, for the maximum bending moment, $M_m = M_\infty = b\sigma_y h^2$, the compressive stress $\sigma_c$ obtained by superposition at $x_3 = h$ remains larger than $-\sigma_y$.

$$
\sigma_c = \sigma_y - (3b\sigma_y h^2/2bh^3)h = -\sigma_y/2
$$

Figure 3: $\sigma_{11}$ stress profile after unloading: (a) after an elastoplastic preloading, (b) when the preloading reaches the limit load

5. Restarting the resolution of the problem with an horizontal force $P$ in addition to the bending moment, the question is to define in the $P$–$M$ plane the “elastic domain” of the system, characterizing the values of the couples $(P, M)$ at the onset of plasticity, and the “limit load”, corresponding to the failure of the structure.

If an horizontal force is applied in addition to the bending moment, the shape of the stress tensor is the same (component $\sigma_{11}$ only), but the neutral line moves. The value is just:

$$
\sigma_{11} = M x_3/I + P/2bh
$$

The boundary of the elastic domain is then a segment in each quadrant of the plan $P$–$M$. The limit load is already defined for pure bending. Without any bending moment, the limit load in tension is nothing but the load that initiate plastic flow:

$$
P_\infty = P_e = 2bh\sigma_y
$$
The combined values of the set \((P_r, M_r)\) leading to the failure of the beam can be found directly, since they do not depend on the loading path. The expressions of the moment and of the horizontal force write in this case (Fig.4a):

if \(x_3 < a\) : \(\sigma_{11} = -\sigma_y\)

if \(x_3 > a\) : \(\sigma_{11} = \sigma_y\).

It comes then:

\[
P = \int_{-h}^{a} -\sigma_y b \, dx_3 + \int_{a}^{h} \sigma_y b \, dx_3 = -2\sigma_y ab
\]

\[
M = \int_{-h}^{a} -\sigma_y x_3 b \, dx_3 + \int_{a}^{h} \sigma_y x_3 b \, dx_3 = b\sigma_y (h^2 - a^2)
\]

After dividing respectively by \(P_e\) and \(M_e\), \(P_r = -P_e a/h\); \(M_r = 3M_e(1 - a^2/h^2)/2\), the resulting diagram is shown in Fig.4b:

\[
M_r/M_e = (3/2)(1 - (P_r/P_e)^2)
\]

6. Calculate the deflection during the loading, and the residual deflection.

Assuming that a plane section remains plane, the displacement field in the beam can be deduced from three variables:

\(u_1 = U(s) + \theta x_3\) \(U(s)\) is the horizontal displacement of the section,

\(\theta\) is the rotation angle around \(x_2\),

\(u_3 = V(s)\) \(V(s)\) is the vertical displacement.

The angle is the opposite of the derivative of the deflection with respect to \(x_1\) (this is needed to have a zero 13 shear strain):

\[
\theta + V_1 = 0
\]

The axial strain can be deduced from the rotation, since

\[
\varepsilon_{11} = u_{1,1} = \theta_{,1} x_3
\]

For the case of an elastic behaviour, the term \(\theta_{,1}\) can be expressed as a function of the applied load

\[
M = \int_{-h}^{+h} \sigma_{11} x_3 b \, dx_3 = EI\theta_{,1}
\]

For the case of a perfectly plastic behaviour, the rotation kinematics is imposed by the elastic zone, so that the orientation is prescribed by the slope between \(-a\) et \(a\). In this area:

\[
\varepsilon_{11} = \sigma_{11}/E = (\sigma_y/E)(x_3/a)
\]
\[ \theta_{1} = \sigma_{y}/Ea \]

Consequently, the curvature \( V_{11} \) is:
- for the elastic regime: \( V_{11} = -M/EI \)
- for the elastoplastic regime: \( V_{11} = -\sigma_{y}/Ea \)

Integrating these two equations for a beam of length \( 2L (-L \leq x_{1} \leq L) \) that is simply supported at its two ends leads to the following expression for the vertical displacement:
- for the elastic regime: \( V = (M/2EI)(L^{2} - x_{1}^{2}) \)
- for the elastoplastic regime: \( V = (\sigma_{y}/2Ea)(L^{2} - x_{1}^{2}) \)

The maximum value of the deflection is obtained for \( x_{1} = 0 \). After replacing \( a \) by its expression (\( a \) is a function of \( M \) in the elastoplastic regime), the global response is obtained:

\[
V = \frac{\sigma_{y}L^{2}}{2Eh\sqrt{3(1 - M/b\sigma_{y}h^{2})}}
\]

Remarks
- This expression is consistent with the expression found for the elastic case when \( M = M_{c} = (2/3)b\sigma_{y}h^{2} \).
- The deflection tends to infinity when \( M \) tends toward \( M_{\infty} = b\sigma_{y}h^{2} \). In this last case, the small perturbation assumption no longer works, far from failure, so that the computation should introduce terms coming from the non linear geometrical evolution.

The residual displacement can be calculated by superposing to the previous expression (obtained for the moment \( M_{m} \)) the displacement field obtained by means of an elastic calculation with a \( -M_{m} \) applied moment, so that:

\[
V = \frac{\sigma_{y}L^{2}}{2Eh\sqrt{3(1 - M_{m}/b\sigma_{y}h^{2})}} - \frac{M_{m}L^{2}}{2EI}
\]