Plasticity criteria

Comparaison of von Mises and Tresca criteria

Assuming that the third eigenstress \( \sigma_3 \) is null, plot the boundary of the elastic domain using von Mises and Tresca criteria.

(a) in a tension-shear plane (only \( \sigma_{11} \) and \( \sigma_{12} \) are non zero); 
(b) in a biaxial tension plane (only \( \sigma_1 \) and \( \sigma_2 \) are non zero).

\[
\begin{align*}
\tau_m &= \frac{\sigma_y}{\sqrt{3}}, \\
T &= \frac{\sigma_y}{2},
\end{align*}
\]

\[
J = \left\{ (3/2)s_i s_j \right\}^{0.5} = \left\{ (1/2) \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \right\}^{0.5}
\]

The criterion is not modified by the addition of a spherical tensor. The shape of the boundary defined by the criterion for the three eigenstresses \((\sigma_1, \sigma_2, \sigma_3)\) is the same as for \((\sigma_1 - \sigma_3, \sigma_2 - \sigma_3, 0)\). The shape in the plane \((\sigma_1 - \sigma_2)\) is then obtained by a simple translation in the direction of the first bissectrice. This result, illustrated in figure ?? for the von Mises case, is also valid for Tresca criterion.

Crystal plasticity

\textit{a. Show that slip on a crystallographic plane is the source of an isochoric strain tensor.}

Let us consider a point \( M(x_1, x_2, x_3) \) in the plane of normal \( n \). The distance between this plane and the origin is \( OP = h \). The question is to evaluate the uniform strain tensor produced by a slip \( \gamma \) along a vector \( m \) belonging to this plane. The displacement value is:

\[
u = (\gamma h) m
\]

Since \( h = \overrightarrow{OM}. n = x_i n_i \):

\[
u_i = \gamma x_i n_i m_i
\]

\[
u_{i,j} = \gamma x_{k,j} n_k m_i \text{ et } u_{j,i} = \gamma x_{k,i} n_k m_j
\]

Comme \( x_{i,j} = \delta_{ij} \):

\[
2\varepsilon_{ij} = u_{i,j} + u_{j,i} = \gamma (n_j m_i + n_i m_j)
\]
The volume change associated to this tensor is zero, since:
\[ \text{trace}(\varepsilon) = n_i m_i = 0 \]

**b. Demonstrate the so called “principle of maximum power” for a material that deforms according to Schmid law.**

For a single plastic slip, defined by the normal vector and the slip direction \((n, m)\), the plastic strain rate writes (with a positive \(\dot{\gamma}\); an external load in the reverse direction would activate an other slip system defined by a direction \(-m\)):

\[ \varepsilon^p = \frac{1}{2} \dot{\gamma} (m \otimes n + n \otimes m) \]

The stress vector on the facet \(n\) is \(T = \sigma : n\), and the resolved shear \(\tau\) in the plane \(n\) in direction \(m\) is:

\[ \tau = m \cdot \sigma \cdot n \]
Due to the symmetry of the stress tensor:

\[ \sigma : \dot{\varepsilon}^p = \frac{1}{2} \dot{\gamma} (n_j m_i + n_i m_j) \sigma_{ij} = \frac{1}{2} \dot{\gamma} n_i m_j = \tau \gamma \]

For an admissible stress state \( \sigma^* : \dot{\varepsilon}^p = \tau^* \gamma \)

The tensor \( \sigma^* \) is admissible according to Schmid law iff the resolved shear stress remains smaller than the critical resolved shear stress, \( \tau_c \). Consequently, using (with \( \gamma > 0 \))

\[ \tau^* \gamma \leq \tau_c \gamma \]

it comes:

\[ \sigma^* : \dot{\varepsilon}^p \leq \sigma : \dot{\varepsilon}^p \]

This result can be generalized to the case of several active slip systems.

**Plastic flow in a thin tube**

_A thin tube has a circular section, with a radius \( r \) and a width \( e \). It is loaded by an internal pressure \( p \). The material is elastic perfectly plastic, with a yield stress \( \sigma_y \). The question is to define the pressure \( P_e \) at the onset of plasticity, and to the direction of the plastic strain rate at the same point. Both Tresca and von Mises criteria will be used, for the three following cases:_

_a. The tube is free in \( z \) direction._
b. The displacement is blocked in \( z \) direction._
c. The tube has a cap (cylindrical tank)._ 

For all the cases, the stress tensor is diagonal in the cylindrical frame \((r, \theta, z)\). Moreover, the stress \( \sigma_{rr} \) can be neglected. It is assumed that both criteria are calibrated to provide the same result in pure tension. This leads to the following expressions, introducing the tensile yield stress \( \sigma_y \):

– von Mises :
  \[ J = \sigma_y \]
– Tresca :
  \[ \max_{i,j} |\sigma_i - \sigma_j| = \sigma_y \]

– For a free tube in \( z \) direction, the tensor is reduced to the diagonal :
  \[ \sigma = \text{Diag}(0; pr/e; 0) \].
– If the axial strain is zero, the expression of \( \varepsilon_{zz} = 0 \) gives :
  \[ \sigma = \text{Diag}(0; pr/e; \sqrt{pr/e}) \].
– Finally, for the case of a tank, the force resulting from the pressure effect on the cap \( (p\pi r^2) \) must be balanced by the stress component \( \sigma_{zz} \) integrated on the section of the tube, so that
  \[ \sigma = \text{Diag}(0; pr/e; pr/2e) \].

_a. The first state of stress is just pure tension. The criteria of von Mises and Tresca predict that plasticity starts on the same point of the loading, when the pressure reaches the value:_

\[ P_e = \frac{\sigma_y e}{P} \]

For the von Mises criterion, the flow direction, defined by the stress deviator, is \( \text{Diag}(-0.5; 1; -0.5) \). The operating point is exactly at an edge of the surface defined by Tresca criterion, written either as

\[ \sigma_{60} - \sigma_{rr} = \sigma_y \text{ or } \sigma_{60} - \sigma_{zz} = \sigma_y \].

The first definition leads to a direction \( \text{Diag}(-1; 1; 0) \), the second \( \text{Diag}(0; 1; -1) \).

_b. This new stress state introduces an intermediate stress \( \sigma_{zz} \) between \( \sigma_{rr} \) and \( \sigma_{60} \). This stress does not play any role in Tresca criterion, and the result is the same as previously. The stress deviator writes:
The von Mises criterion predicts that plasticity occurs for:

\[ P_e = \frac{\sigma_y e}{r\sqrt{1-\nu+\nu^2}} \]

Using \( \nu = 0.3 \), the pressure value is then \( 1.125\sigma_y e/r \); the von Mises criterion is then “optimistic”.

The direction of the plastic flow for the von Mises criterion is still proportional to the deviator. For Tresca’s criterion, the plastic flow is well defined by \( \text{Diag}(-1;1;0) \) since now \( \sigma_{00} - \sigma_{rr} = \sigma_y \) is the only valid expression.

c. Tresca’s criterion is unchanged again. The deviator and the limit pressure according to von Mises criterion are now: \( (pr/3e)\text{Diag}(-0.5;0.5;0) \), et:

\[ P_e = \frac{2\sigma_y e}{r\sqrt{3}} \]

that is \( 1.15\sigma_y e/r \).

For both criteria, the flow direction is now a pure shear, with a zero \( z \) component. Figure 3 illustrates the various stress states in the plane \( \sigma_{00} - \sigma_{zz} \), and the predicted flow directions.

**Tresca’s criterion**

*Find the equivalent strain associated to Tresca’s criterion.*

Using the eigenstresses, the expression of Tresca’s criterion is

\[ \max_{i,j} |\sigma_i - \sigma_j| = \sigma_y \]

Assuming that \( \sigma_1 > \sigma_2 > \sigma_3 \), it becomes \( \sigma_1 - \sigma_3 = \sigma_y \).

The plastic flow is defined by the diagonal \( \text{Diag}(0.5;0;0.5)\dot{\lambda} \).

If two eigenstresses are equal, for instance \( \sigma_1 = \sigma_2 > \sigma_3 \), two candidate expressions can be written, with two plastic multipliers. The plastic strain rate writes then (using a scalar \( k \) between -1 and +1) (the notation \( \dot{\lambda} \) is not the same in the two equations):

\[ \dot{\varepsilon}^p = \text{Diag}(\dot{\lambda};\dot{\mu};-(\dot{\lambda} + \dot{\mu})) \]
\[ \varepsilon'' = \text{Diag}(0.5(1+k);0.5(1-k);-1) \dot{\lambda} \]

Similar expressions can be obtained for different combinations of the eigenstresses. It can be observed, for all the cases, that:

\[ \dot{\lambda} = \frac{1}{2} \left( |\varepsilon''_1| + |\varepsilon''_2| + |\varepsilon''_3| \right) \]